UNIT - I

Systems and Representation

Basic elements in control systems :- open & closed Loop systems - Electrical analogy of Mechanical and thermal systems - Transfer function - Ac & Dc Servomotors - Block diagram Reduction Techniques - Signal flow graph.

System :-

When a number of elements (or) component are connected in a sequence to perform a specific function, the group thus formed is called System.

Control System :-

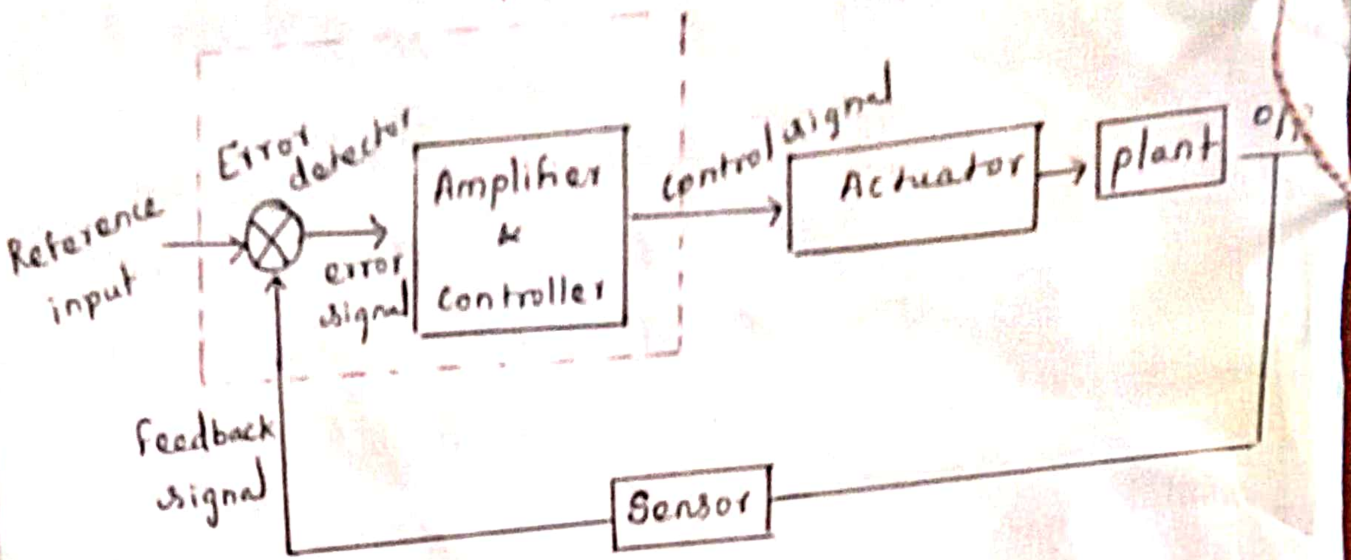
In a system, when the output quantity is controlled by varying the input quantity the system is called control system.

(or)

A control system is a collection of physical object (components) connected together to serve an objectives. The input & output relation of various physical components are governed by differential equations.

Basic elements in control ...

Automatic control System



The basic elements in control systems are

→ plant

→ controller & Amplifier

→ Actuator

→ Error detector

→ Sensor (feed back)

plant :

- plant is open loop system
- The output is automatically controlled by means of closed loop system.

Reference input :

- Reference input signal corresponds to desired output.

Feed back signal :-

- The feedback signal is proportional to the output signal

· It is fed to the error detector.

Error detector:-

→ The error signal is generated by the error detector.

→ The error signal is the difference between the reference signal & feedback signal.

Amplifier & controller:-

→ It modify and amplifies the error signal for better control actions.

Actuator: →

→ power Amplifying devices.

→ produce input to the plant according to the control signal.

Sensor :-

→ It will sense the output & given desired feedback signal to the error detector.

→ Transducer is act as sensor. It will convert non electrical quantity to electrical quantity.

- It is fed to the error detector.

Error detector :-

→ The error signal is generated by the error detector.

→ The error signal is the difference between the reference signal & feedback signal.

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→ It will sense the output & give desired feedback signal to the error detector.

→ Transducer is act as sensor. It will convert non electrical quantity to electrical quantity.

Open Loop and closed Loop System

In a system, when the output quantity is controlled by varying the input quantity, the system is called control system.

Output quantity → controlled variable (or) response.

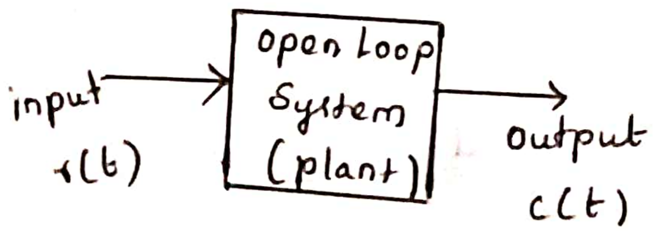
input quantity → command signal (or) excitation.

Open Loop System :-

Any physical system which doesn't automatically correct the variation in its output is called open Loop system.

ie) o/p quantity has no effect on input quantity.

→ No feedback



→ The output can be varied by varying the input.

→ Due to external disturbances, the system output changes.

→ the changes in output are corrected by changing the input manually.

Advantage of open Loop System:

→ Simple & economical

→ Easier to construct

→ System is stable.

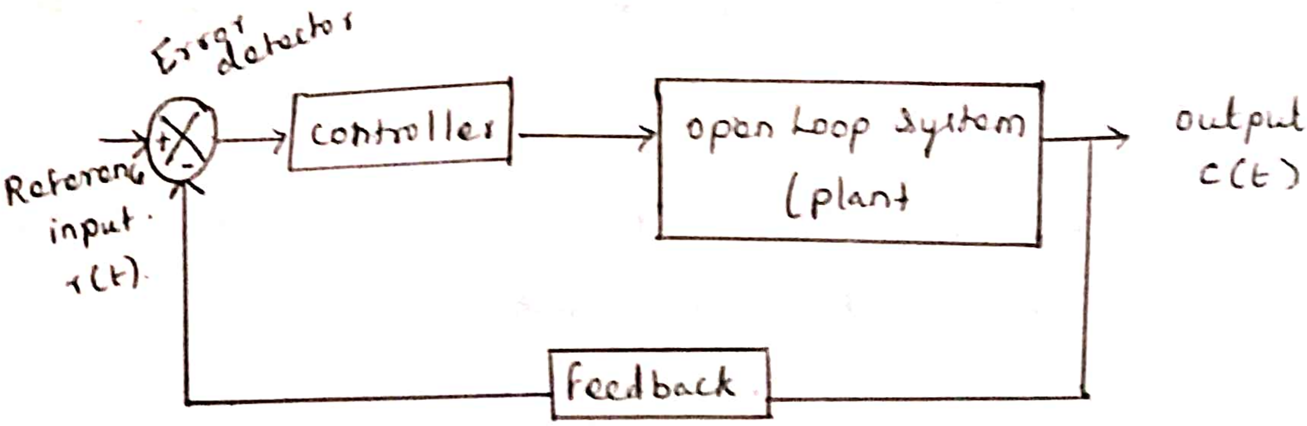
Dis Advantage :-

→ Inaccurate & unreliable

→ changes in the output due to external disturbances are not corrected automatically.

Closed Loop System.

The control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed Loop system.



→ The open Loop system can be modified as closed Loop system by providing a feedback.

→ The feedback automatically corrects the changes in output due to disturbances.

→ the closed Loop system is called as (Automatic control system).

→ the reference signal (or) input signal corresponds to desired output.

→ The feedback path elements samples the output and converts in to a signal of same type as that of reference signal.

Feed back signal α output signal.

→ the ~~output~~ feed back signal is fed to the error detector. the error signal is generated by error detector. It is the differaence between reference signal & feedback signal.

→ The controller modifies & amplifies the error signal to better control action.

→ the modified error signal is fed to the plant to correct its output.

Advantage :-

→ Accurate

→ Less affected by noise.

disadvantage :-

→ complex & costly

→ feed back may lead to oscillation

→ overall gain is reduced

→ Stability is the major problem

Electrical Analogy of Mechanical Systems & Thermal Systems.

System remain analogous as long as the differential equations governing the system (or) Transfer functions are identical form.

Electrical System

Resistor, Inductance & capacitance → basic elements of electric circuit
(R) (L) (C).

inputs → voltage source (or) current source.

Mechanical System:

A device can be modelled by
→ Translatory
→ Rotary elements.

Mass, Dash pot, Spring → Translational element.
(M) (B) (K)

moment of Inertia, Rotational dashpot, Torsional spring.
(J) (B) (K)

Translatory → Rotational elements.

input → Force → Torque.

output → velocity. → Angular velocity

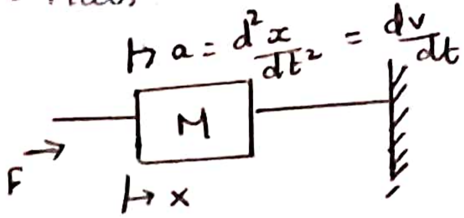
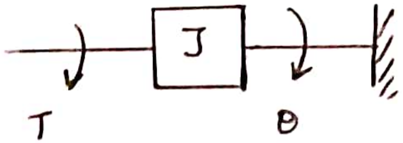
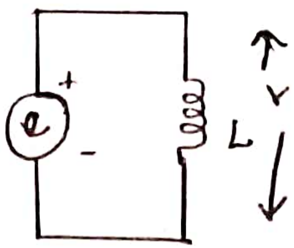
There are two types of Analogies.

Mechanical Translatory → Force-voltage analogy
→ Force-current analogy.

Mechanical rotational → Torque-voltage Analogy
→ Torque-current analogy

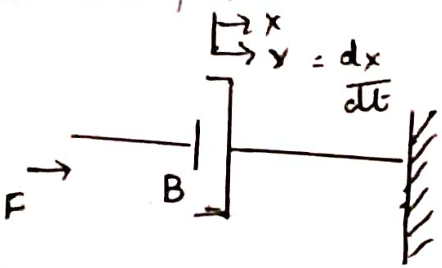
Electrical analogous of Mechanical Translational System & Rotational systems.

(i) Force (Torque) - Voltage Analogy.

Items	Mechanical Translational System	Mechanical rotational System	Electrical System
Input	Force (F)	Torque (T)	voltage (v, e)
output	velocity (v) displacement (x)	angular velocity (ω) angular displacement (θ)	current (i)
Elements.	M - Mass $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$  $F = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	Moment of inertia (J)  $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$	Inductance, L. $e = v$, $v = L\frac{di}{dt}$  $e = L\frac{di}{dt}$

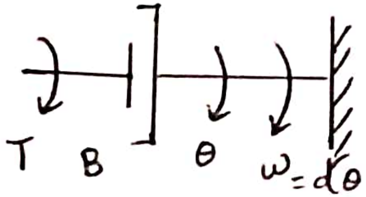
→ Stability is the major ...

B - Dashpot.



$F = B\frac{dx}{dt} = Bv$

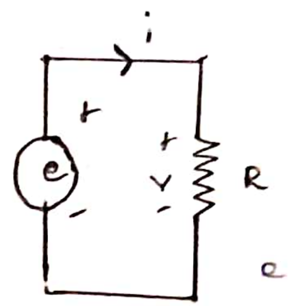
B - Rotational dashpot



$\omega = \frac{d\theta}{dt}$.

$T = B\frac{d\theta}{dt} = B\omega$.

Resistance.

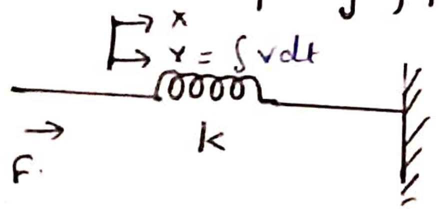


$e = v$

$v = iR$

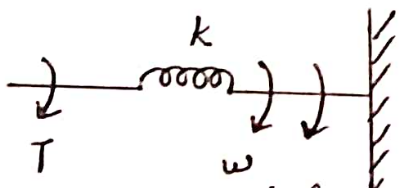
$e = Ri$

Stiffness of Spring, K



$F = Kx = K\int v\,dt$

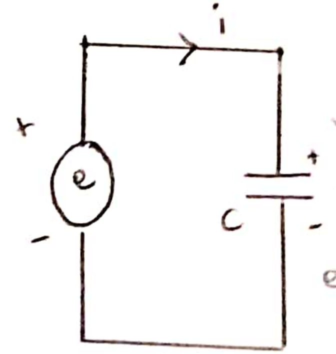
K → Stiffness of spring



$\theta = \int \omega\,dt$

$T = K\theta = K\int \omega\,dt$.

capacitance -



$e = v$

$v = \frac{1}{C}\int i\,dt$

$e = \frac{1}{C}\int i\,dt$

physical Law.

Newton's 2nd Law

$\Sigma F = 0$

Newton's second Law

$\Sigma T = 0$

Kirchoff's voltage Law

$\Sigma V = 0$

changing the Level of independent Variable

Lever

$\frac{F_1}{F_2} = \frac{l_1}{l_2}$

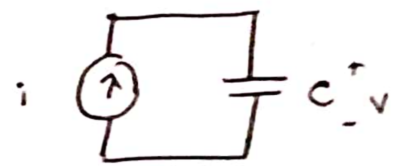
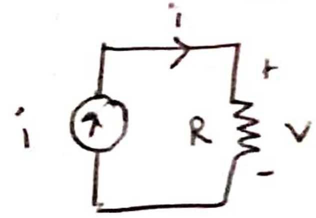
Gear

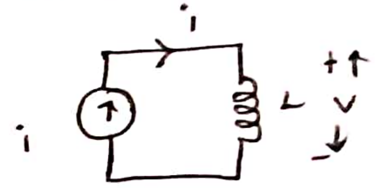
$\frac{T_1}{T_2} = \frac{n_1}{n_2}$

Transformer.

$\frac{e_1}{e_2} = \frac{N_1}{N_2}$

(ii) Force (Torque) - current Analogy .

Items	Mechanical Translational System	Mechanical rotational System	Electrical System
Input	Force (F)	Torque (T)	current (i)
Output	Velocity (v) displacement (x)	angular velocity (ω) angular displacement (θ)	voltage (v)
Elements .	M - Mass . $F = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$ B - dashpot $F = B\frac{dx}{dt} = Bv$	J - moment of inertia $F = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ B - rotational dashpot $T = B\frac{d\theta}{dt} = B\omega$.	Capacitance .  $i = C\frac{dv}{dt}$ Conductance  $i = \frac{1}{R}V$

	K - Spring of stiffness. $F = kx = k\int v\,dt$.	(torsional spring) K - rotational spring of stiffness. $T = k\theta = k\int \omega\,dt$.	Inverse of inductance  $i = \frac{1}{L}\int v\,dt$
physical Law.	Newton's 2^{nd} Law. $\Sigma F = 0$	Newton's 2^{nd} Law $\Sigma T = 0$	Kirchoff's ~~voltage~~ current Law $\Sigma i = 0$
changing the Level of Independent Variable.	Lever $\frac{F_1}{F_2} = \frac{l_1}{l_2}$	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer. $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

M (mass) → weight of Mechanical System.

K (spring) → elastic deformation of a body.

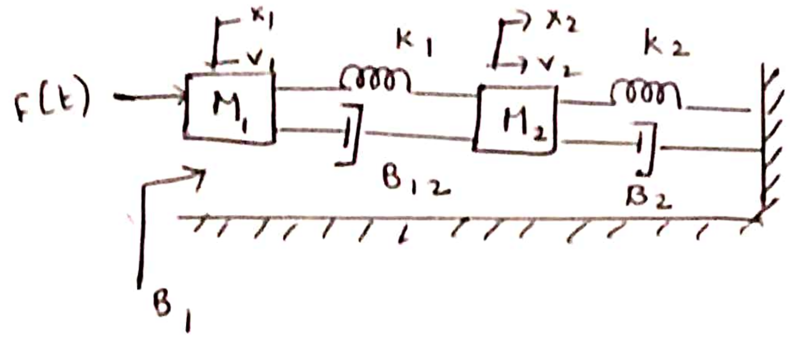
B (dashpot) → friction exist in ~~rotating~~ Mechanical system

||^{rly} J (Mass) → weight of rotational mechanical system
(moment of inertia)

K (Torsional spring) → elastic deformation of body

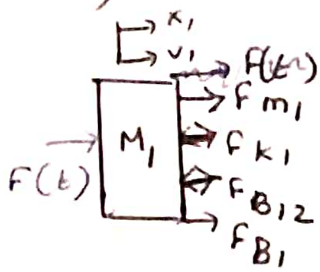
B (dash pot) → friction exist in rotational mechanical system problem

1) write the differential equations governing the Mechanical system shown in fig. draw the force-voltage & force-current analogous circuit & verify by writing mesh & node equations.



Solution:

Free body diagram for mass m_1,



$$f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{B_1} = B_1 \frac{dx_1}{dt}$$

$$f_{k_1} = k_1 (x_1 - x_2).$$

$$f_{B_{12}} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

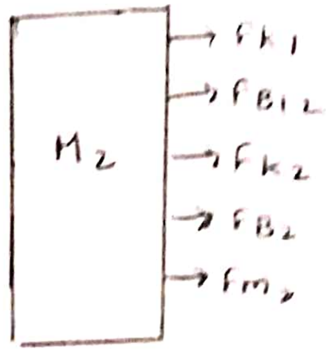
By newton's 2nd Law.

$$\Sigma f = 0$$

$$f_{m_1} + f_{B_1} + f_{k_1} + f_{B_{12}} = f(t).$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + k_1(x_1 - x_2) + B_{12}\frac{d}{dt}(x_1 - x_2) = f(t) \rightarrow (1)$$

Free body diagram for mass m_2



$$f_{m_2} = M_2 \frac{d^2 x_2}{dt^2}$$

$$f_{B_{12}} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k_1} = k_1 (x_2 - x_1)$$

$$f_{k_2} = k_2 x_2$$

$$f_{B_2} = B_2 \frac{d x_2}{dt}$$

By newton's 2^{nd} Law $\Sigma f = 0$

$$f_{m_2} + f_{B_{12}} + f_{k_1} + f_{k_2} + f_{B_2} = 0$$

$$M_2 \frac{d^2 x_2}{dt^2} + B_{12} \frac{d}{dt}(x_2 - x_1) + k_1 (x_2 - x_1) + k_2 x_2 + B_2 \frac{d x_2}{dt} = 0 \rightarrow (2)$$

equation (1) & (2) are replaced by velocity (v)

$$\frac{d^2 x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v, \quad x = \int v\,dt$$

$$(1) \Rightarrow M_1 \frac{dv}{dt} + B_1 v + k_1 \int (v_1 - v_2)\,dt + B_{12}(v_1 - v_2) = f(t) \rightarrow (3)$$

$$(2) \Rightarrow M_2 \frac{dv}{dt} + B_{12}(v_2 - v_1) + k_1 \int (v_2 - v_1)\,dt + k_2 \int v_2\,dt + B_2 v_2 = 0 \rightarrow (4)$$

Force - voltage Analogous.

$f(t) \rightarrow e(t)$, $k_1 = 1/C_1$, $k_2 = 1/C_2$.

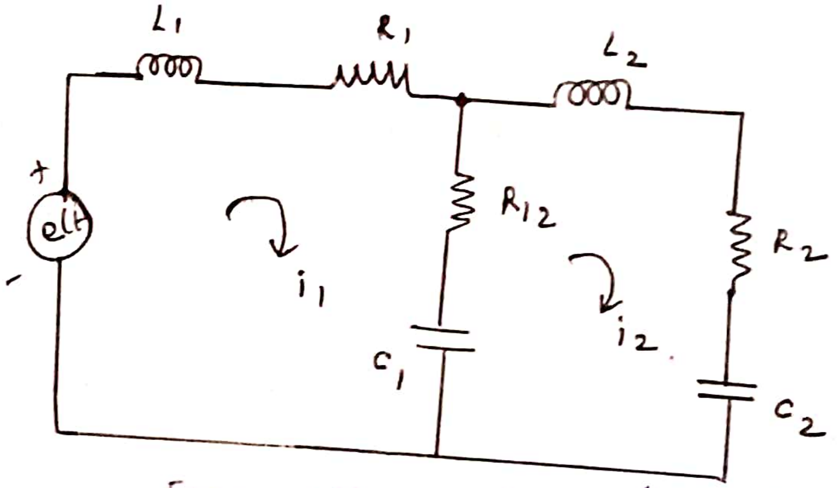
$M_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad v_1 \rightarrow i_1$

$M_2 \rightarrow L_2$ $\quad B_2 \rightarrow R_2$ $\quad v_2 \rightarrow i_2$.

(3) ⇒

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int (i_1 - i_2)\,dt + R_{12}(i_1 - i_2) = e(t)$$

(4) ⇒ $$L_2 \frac{di_2}{dt} + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt + \frac{1}{C_2}\int i_2\,dt + R_2 i_2 = 0.$$



Force - voltage electrical analogous circuit

Force - current Analogous circuit

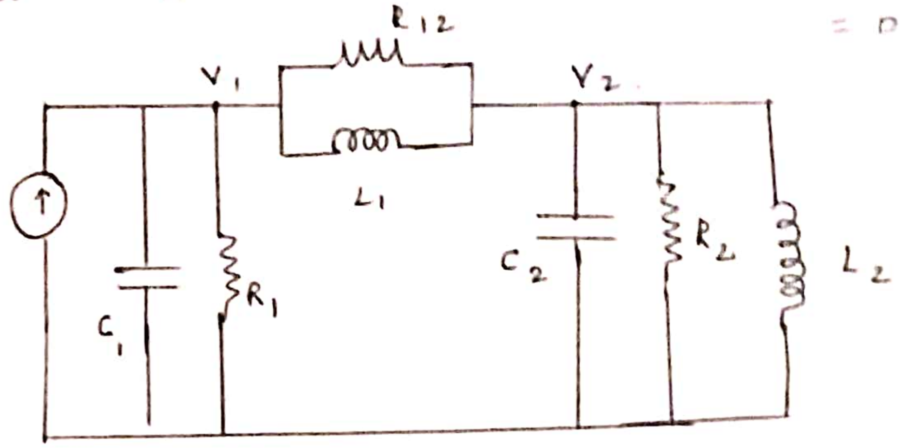
$f(t) \rightarrow I(t)$ $\quad M_1 \rightarrow C_1$, $B_1 \rightarrow 1/R_1$, $k_1 \rightarrow 1/L_1$

$v_1 \rightarrow V_1$ $\quad M_2 \rightarrow C_2$ $\quad B_2 \rightarrow 1/R_2$ $\quad k_2 \rightarrow 1/L_2$

$v_2 \rightarrow V_2$ $\quad B_{12} \rightarrow 1/R_{12}$

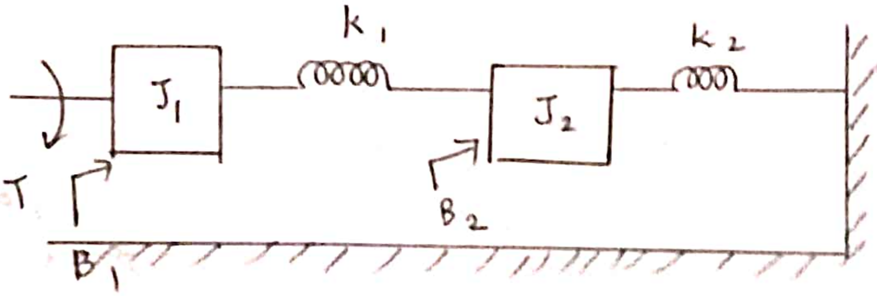
(3) $\Rightarrow C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{R_{12}}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\, dt = i(t)$

(4) $\Rightarrow C_2 \frac{dv_2}{dt} + \frac{1 v_2}{R_2} + \frac{1}{L_2}\int v_2\, dt + \frac{1}{R_{12}}(v_2 - v_1) + \frac{1}{L_1}\int (v_2 - v_1)\, dt = 0$



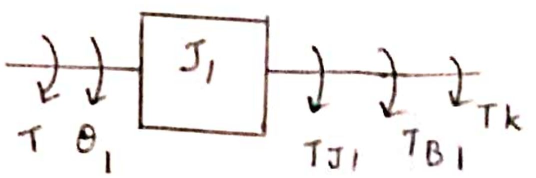
problem 2:

2) write the differential equations governing the Mechanical rotational System shown. draw the Torque - voltage & torque current electrical analogous circuit.



Solution:-

The free body diagram of J_1



$T_{J1} = J_1 \frac{d^2\theta_1}{dt^2}$

$T_{B1} = B_1 \frac{d\theta_1}{dt}$

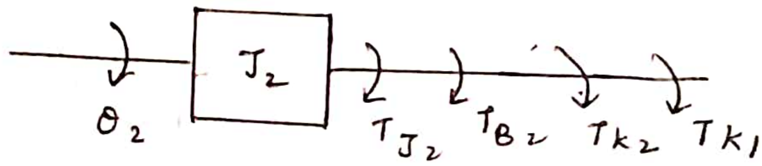
$T_{k1} = k_1(\theta_1 - \theta_2)$

By newton's 2nd Law $\Sigma T = 0$.

$$T = T_{J_1} + T_{B_1} + T_{k_1}$$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + k_1 (\theta_1 - \theta_2) = T \rightarrow (1)$$

free body diagram of J_2.



$$T_{J2} = J_2 \frac{d^2\theta_2}{dt^2}$$

$$T_{B2} = B_2 \frac{d\theta}{dt}$$

$$T_{K1} = k_1 (\theta_2 - \theta_1).$$

$$T_{k_2} = k_2 \theta_2$$

By Newton's 2nd Law $\Sigma T = 0$.

$$T_{J_2} + T_{B_2} + T_{K_2} + T_{K2} = 0$$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + k_2\theta_2 + k_1(\theta_2 - \theta_1) = 0 \rightarrow (2)$$

Replace $\frac{d^2\theta}{dt^2} = \frac{dw}{dt}$, $\frac{d\theta}{dt} = w$, $\theta = \int w dt$.

$$(1) \Rightarrow J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + k_1 \int(\omega_1 - \omega_2)dt = T \rightarrow (3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + k_2 \int \omega_2 dt + k_1 \int(\omega_2 - \omega_1) dt = 0 \rightarrow (4)$$

Torque - voltage Analogous circuit.

$T \rightarrow e(t)$ $\quad J_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_1 \rightarrow 1/C_1$

$\omega_1 \rightarrow i_1$ $\quad J_2 \rightarrow L_2$ $\quad B_2 \rightarrow R_2$ $\quad K_2 \rightarrow 1/C_2$

$\omega_2 \rightarrow i_2$

$$(3) \Rightarrow L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int(i_1 - i_2) = e(t)$$

$$(4) \Rightarrow L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int(i_2 - i_1)dt = 0$$

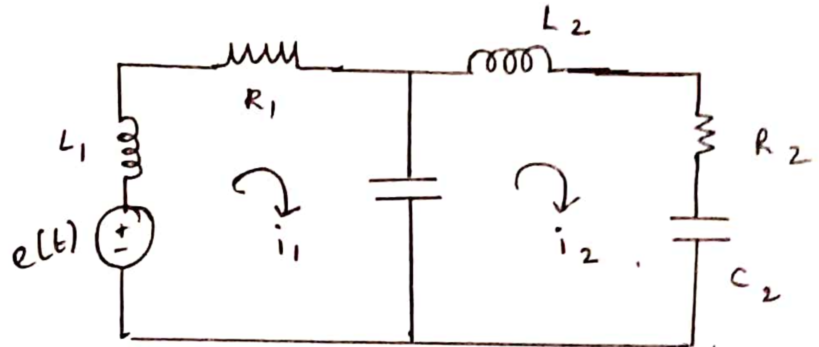


fig: Torque - voltage analogous.

Torque - current Analogous circuit

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad J_1 \rightarrow C_1$ $\quad k_1 \rightarrow 1/L_1$

$\omega_1 \rightarrow v_1$ $\quad B_2 \rightarrow 1/R_2$. $\quad J_2 \rightarrow C_2$ $\quad k_2 \rightarrow 1/L_2$

$\omega_2 \rightarrow v_2$

$$(3) \Rightarrow C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1}\int(v_1 - v_2)\, dt = e(t)$$

$$(4) \Rightarrow C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{L_1}\int(v_2 - v_1)dt = 0$$

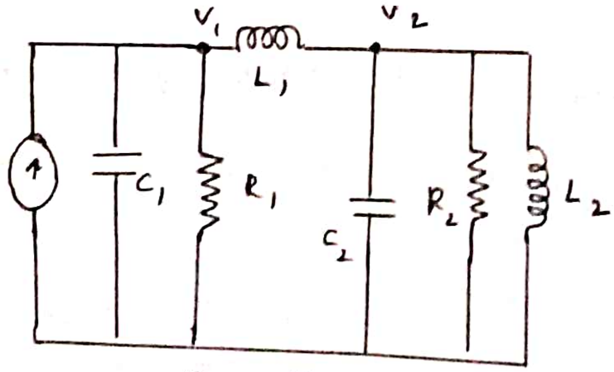


Fig. Torque - current analogous.

Thermal System :-

For Analysis purpose, Assume the thermal System of the temperature of the medium is uniform.

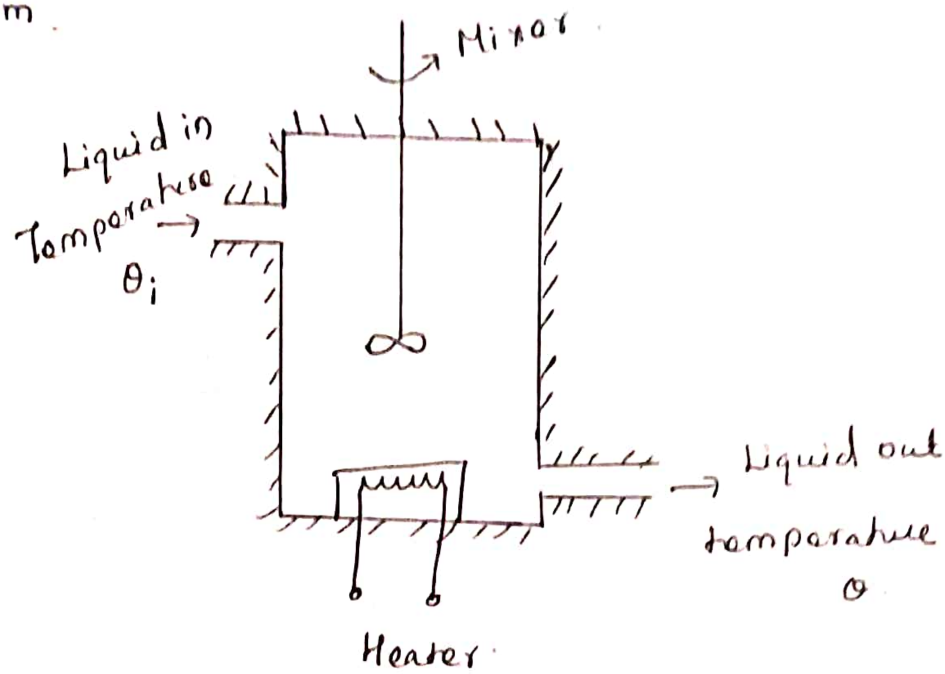


fig: thermal System.

Assume,

→ Tank is insulated to eliminate heat Loss to the surrounding air.

→ Liquid in the tank is kept at uniform temperature with the help of a stirrer by perfect mixing.

→ Steady state temperature of inflowing Liquid is θ_i & outflowing Liquid is θ.

H → Steady state heat input rate from the heater.

→ Liquid flow rate is constant.

Let

$\Delta H \rightarrow$ small increase in the heat input rate from its steady state value

$\Delta H_1 \rightarrow$ heat outflow rate by an

$\Delta H_2 \rightarrow$ heat storage liquid in the tank

$\Delta \theta \rightarrow$ outflowing liquid rises

Heat outflow rate is

$$\Delta H_1 = Qs\,\Delta\theta \rightarrow (1)$$

Where, $Q \rightarrow$ steady liquid flow rate in kg/min

$s \rightarrow$ specific heat of the liquid in J/kg°C

Thermal Resistance $R = 1/Qs$ (°C/J/min) $\rightarrow (2)$

sub eq (2) in eq (1)

$$\Delta H_1 = \frac{\Delta\theta}{R} \rightarrow (3)$$

The rate of heat storage in the tank is given by

$$\Delta H_2 = Ms\frac{d(\Delta\theta)}{dt} \rightarrow (4)$$

Where, $M \rightarrow$ mass of liquid in the tank kg

$\frac{d(\Delta\theta)}{dt} \rightarrow$ rate of rise of temperature in the tank

Thermal capacitance $C = Ms$ (J/°C) $\rightarrow (5)$

sub (5) in (4)

$$\Delta H_2 = C\frac{d(\Delta\theta)}{dt} \rightarrow (6)$$

Heat Flow Balance equation

$$\Delta H = \Delta H_1 + \Delta H_2$$

$$= \frac{\Delta\theta}{R} + C\frac{d(\Delta\theta)}{dt}.$$

$$\Delta H = \frac{\Delta\theta + RC\frac{d(\Delta\theta)}{dt}}{R}$$

$$R(\Delta H) = \Delta\theta + RC\frac{d(\Delta\theta)}{dt} \rightarrow (i)$$

This equation describes that the dynamic of the thermal systems with the assumption that the temperature of the inflowing liquid is constant.

But practice, the ~~the~~ temperature of the inflowing liquid fluctuates. Thus along with the heat input signal from the heater, there is an additional signal due to change in temperature of the inflowing liquid which is known as disturbance signal.

Let, $\Delta\theta_i \rightarrow$ change in the temperature of the inflowing liquid from its steady state value.

In addition to the change in heat input from the heater, there is a change in heat carried by the inflowing liquid.

The heat flow equation,

$$\Delta H + \frac{\Delta\theta_i}{R} = \frac{\Delta\theta}{R} + C\frac{d}{dt}(\Delta\theta) \rightarrow (8)$$

(or)

$$RC\frac{d}{dt}(\Delta\theta) + \Delta\theta = \Delta\theta_i + R(\Delta H)$$

Assume, the tank insulation is perfect.

As the liquid temperature increased by $\Delta\theta$, the rate of heat flow through the tank walls to the ambient medium increased by

$$\Delta H_3 = \frac{\Delta\theta}{R_t}$$

Where,

$R_t \rightarrow$ thermal resistance of the tank walls.

$$\Delta H + \frac{\Delta\theta_i}{R} = \left(\frac{\Delta\theta}{R} + \frac{\Delta\theta}{R_t}\right) + C\frac{d}{dt}(\Delta\theta)$$

∴ There is no heat storage in the tank wall.

Electrical System	Thermal System
charge q	Heat flow J
current A	Heat flow rate J/min
Voltage V	Temperature °C
Resistance Ω	Resistance °C/(J/min)
Capacitance, C	Capacitance J/°C

Transfer function

The transfer function of a system is defined as the ratio of Laplace Transform of output to the Laplace transform of input with all initial conditions are zero.

$$\text{Transfer function} = \left.\frac{L.T \text{ of output}}{L.T \text{ of input}}\right|_{\text{with zero initial condition}}$$

procedure:

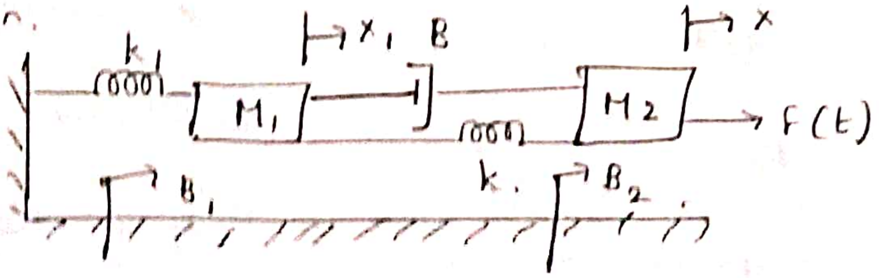
Step 1: write the differential equations of the given system.

Step 2: Take Laplace Transform of the equation and assume all initial condition are zero

Step 3: Take the ratio of transformed output to input for the given system.

Problem

1) write the differential equations governing the Mechanical system. and determine the transfer function.



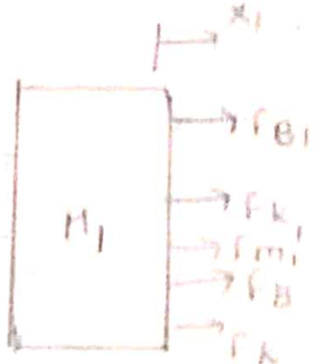
Solution :-

The System has two nodes and they are mass M_1 & M_2.

input $\rightarrow$ $f(t)$

output $\rightarrow$ displacement x.

Step 1: the free body diagram of mass M_1



Let, $f_{m_1} = M_1 \dfrac{d^2 x_1}{dt^2}$

$F_{B_1} = B_1 \dfrac{dx_1}{dt}$

$F_{k_1} = k_1 x_1$

$F_B = B \dfrac{d}{dt}(x_1 - x)$

$F_k = k(x_1 - x)$

By newton's 2nd Law

$$f_{m_1} + f_{B_1} + f_{k_1} + f_B + f_k = 0$$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + k_1 x_1 + k(x_1 - x) = 0.$$

Taking L.T

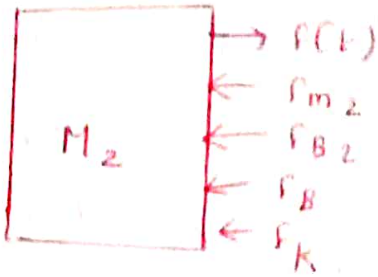
$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + k_1 X_1(s)$$
$$+ k(X_1(s) - X(s)) = 0$$

$$X_1(s)[M_1 s^2 + B_1 s + Bs + k_1 + k] = X(s)[Bs + k] \rightarrow (1)$$

$$X_1(s) = \frac{X(s)[Bs + k]}{M_1 s^2 + B_1 s + Bs + k_1 + k} \rightarrow (2)$$

Step 2:

The free body diagram of mass M_2, the



$$F_{m2} = M_2 \frac{d^2x}{dt^2}, \quad f_{B2} = B_2 \frac{dx}{dt}$$

$$f_B = B \frac{d}{dt}(x - x_1), \quad f_k = k(x - x_1)$$

By newton's 2nd Law,

$$F_{m2} + f_{B2} + f_B + f_k = f(t).$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + k(x - x_1) = f(t)$$

Taking Laplace Transform,

$$M_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + k[X(s) - X_1(s)] = F(s)$$

$$X(s)[M_2 s^2 + (B_2 + B)s + k] - X_1(s)[Bs + k] = F(s) \quad \rightarrow (2)$$

Sub (a) in equation (2)

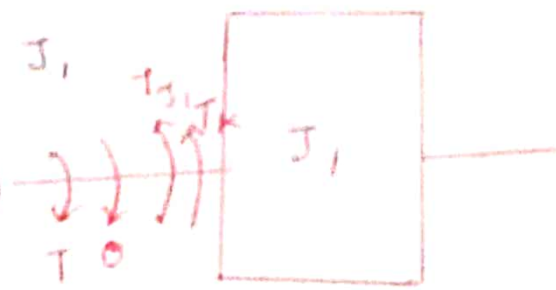
$$(2) \Rightarrow X(s)[M_2 s^2 + (B_2 + B)s + k] - \frac{X(s)(Bs + k^2)}{M_1 s^2 + (B_1 + B)s + (k_1 + k)} = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B_1 + B)s + (k_1 + k)}{[M_1 s^2 + (B_1 + B)s + (k_1 + k)][M_2 s^2 + (B_2 + B)s + k] - (Bs + k)^2}$$

2. Write the differential equations governing the Mechanical rotational system shown in fig. Obtain the transfer function of the system.

Solution

Step 1 Free body diagram of J_1



$$T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}, \quad T_k = k(\theta_1 - \theta)$$

by Newton's 2nd Law $T_{J_1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + k(\theta_1 - \theta) = T$$

Taking Laplace Transform

$$J_1 s^2 \theta_1(s) + k\,\theta_1(s) - k\,\theta(s) = T(s)$$

Step 2: Free body diagram of mass with moment of Inertia J_2

$$T_{J_2} = J_2 \frac{d^2\theta}{dt^2}, \quad T_B = B\frac{d\theta}{dt}, \quad T_k = k(\theta - \theta_1)$$

By Newton's 2nd Law,

$$T_{J_2} + T_b + T_k = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + k(\theta - \theta_1) = 0$$

Taking Laplace Transform

$$J_2 s^2\theta(s) + Bs\,\theta(s) + k\,\theta(s) - k\,\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + k)\,\theta(s) - k\,\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + k)\,\theta(s)}{k} \quad \text{—(3)}$$

Sub eq (3) in eq (1)

$$(J_1 s^2 + k)\,\frac{(J_2 s^2 + Bs + k)}{k}\,\theta(s) - k\,\theta(s) = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{k}{(J_1 s^2 + k)(J_2 s^2 + Bs + k) - k^2}$$

DC Servomotor & AC Servomotor

The motor that are used in automatic control systems are called servomotor.

To control the position of an object in a system is called Servo Mechanism

→ The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft.

→ They can operate in continuous duty or step duty depending upon construction

Servo motor are classified into two type

→ Dc servomotor

→ Ac servomotor

Dc servomotor :-

Dc servomotor are classified into

Dc servomotor

- Sliding contact Motors with commudator & brushes.
  - permanent Magnet Motor
    1. cylindrical armature motors
    2. Disc armature motors
    3. moving coil motors
  - Electro magnetic fields motor
    1. Armature controlled motors
    2. field controlled motors
    3. Series motors
    4. Split field motor
- Brushless or contactless motor with transistor commutator.

permanent Magnet motors :

→ In this motor, field winding is replaced by permanent magnet to produce the required magnetic field.

→ It is more economical for power rating upto a few kW.

→ The armature is placed in rotor.

→ permanent magnet poles are fixed to the Stator.

→ Rotor employs special type of construction to reduce the weight & so inertia of the rotating system

Special type of construction.

cylindrical armature → small diameter & Longer axial length.

disc armature → moving coil motor
&
hollow armature

Advantage.

→ high efficiency.

→ Simple & more reliable.

→ Less heating.

→ high power output.

Disadvantage

→ cost of material are very high

→ magnets deteriorate with time & demagnetised by large current transient

Electromagnetic field motors

→ It is economical for high power rating generally above 1kw.

Special features -

1. The No. of slots & commutator segments is Large to improve commutation.

2. Compoles & compensating windings are provided to eliminate sparking.

3. $D/L \simeq$ Low, to reduce inertia.

4. Oversize shafts are employed to withstand the high torque stress.

5. Eddy current are reduced by complete Lamination of the magnetic circuit and by using Low Loss steel.

T&N are controlled by varying the armature current &/or the field control current.

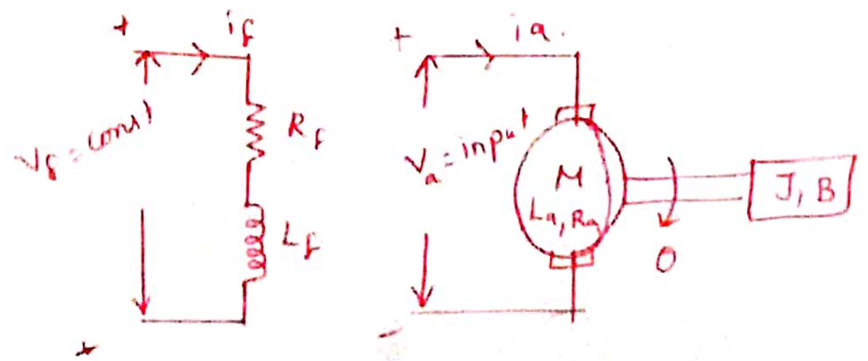
In armature controlled mode → field current is constant & Armature current is varied to control the torque

$V_a, I \rightarrow I_f$ const

In field controlled mode → Armature current is constant & field current is varied to control the torque.

$I_f, I \rightarrow V_a$

Armature controlled dc servomotor.



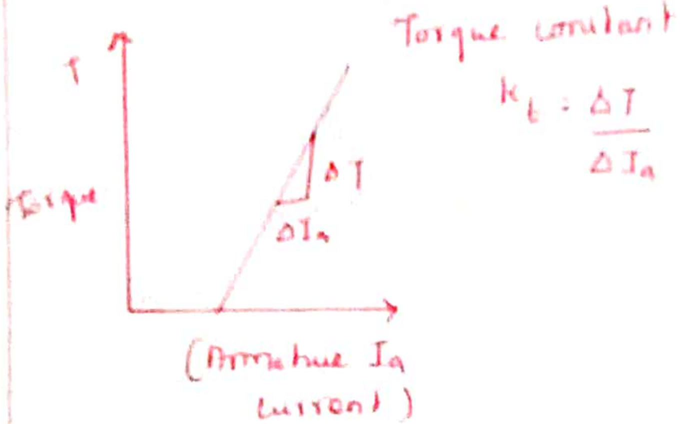
The field is excited by a constant dc supply

If I_f = const, $N \propto V_a$

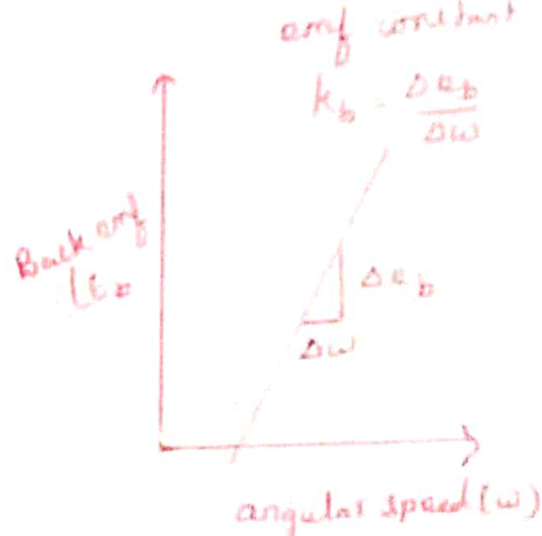
$T \propto I_a$

Torque & speed can be controlled by armature voltage

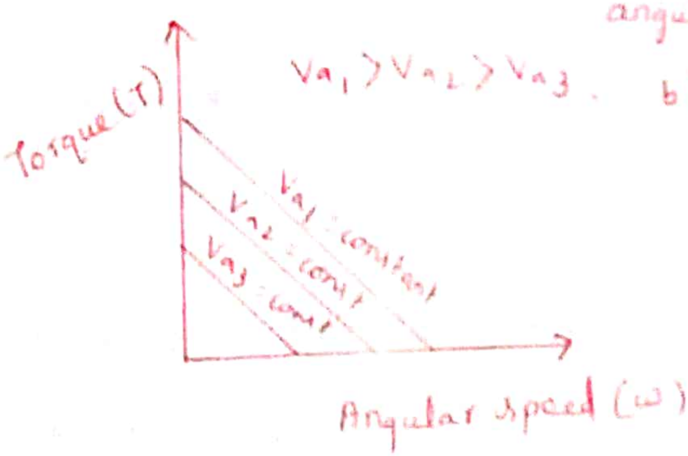
For Reversible operation → Reversing the armature voltage



a) I_a Vs T



b) ω Vs E_b



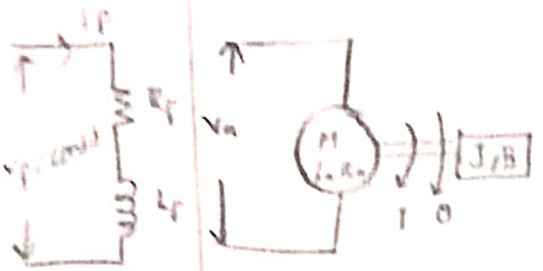
c) ω Vs T

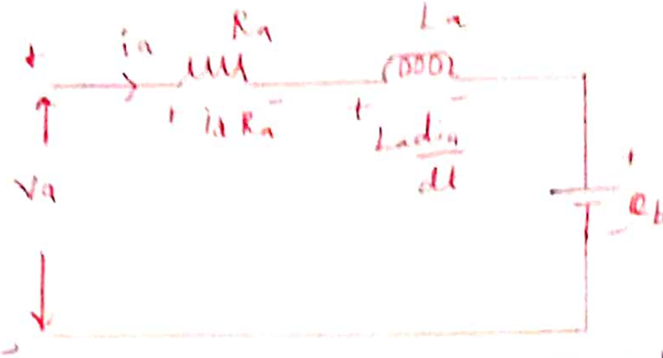
Characteristics of Armature controlled dc servomotor

In small motor, the armature voltage is controlled by variable Resistance.

In Large motor, V_a is controlled by thyristor

Transfer function of Armature controlled by DC motor





equivalent circuit of Armature.

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \rightarrow (1)$$

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \rightarrow (2)$$

Torque is proportional to the product of flux & current.

$T \propto \varphi I_a$. φ = constant.

Torque

$$T \propto i_a$$

$$T = k_t i_a \rightarrow (3)$$

Back emf $\propto$ speed.

$$e_b \propto \frac{d\theta}{dt}$$

$$e_b = k_b \frac{d\theta}{dt} \rightarrow (4).$$

Taking L.T.

$$R_a I_a(s) + L_a s \cdot I_a(s) + E_b(s) = V_a(s) \rightarrow (5)$$

$$T(s) = k_t I_a(s) \rightarrow (6)$$

$$Js^2 \theta(s) + Bs\theta(s) = T(s) \rightarrow (7)$$

$$E_b(s) = k_b s \theta(s) \rightarrow (8)$$

equating (6) & (7)

$$Js^2\theta(s) + Bs\,\theta(s) = k_t I_a(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)\;\theta(s)}{k_t} \rightarrow (9)$$

sub eq (9) (8) in (5)

$$I_a(s)\left[R_a + L_a s\right] + E_b(s) = V_a(s)$$

$$\frac{(Js^2 + Bs)}{k_t}(R_a + L_a s)\,\theta(s) + k_b s\,\theta(s) = V_a(s)$$

$$\theta(s)\left[\frac{(Js^2+Bs)}{k_t}(R_a + L_a s] + k_b s\right] = V_a(s)$$

$$\frac{\theta(s)}{V_a(s)} = \frac{k_t}{[(R_a + L_a s)(Js^2 + Bs) + k_b k_t s]}$$

$$= \frac{k_t}{R_a\left[1 + \frac{L_a}{R_a}s\right] Bs\left[1 + \frac{Js^2}{Bs}\right] + k_b k_t s}$$

Where,

$$\frac{L_a}{R_a} = T_a = \text{electrical time constant}$$

$$J/B = T_m = \text{Mechanical time constant}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t/R_a B}{s\left[(1+sT_a)(1+sT_m) + \frac{K_b K_t}{R_a B}\right]}$$

This is the transfer function of armature controlled dc motor.

Field controlled DC servomotor.

In this motor, the armature is supplied with constant current or voltage.

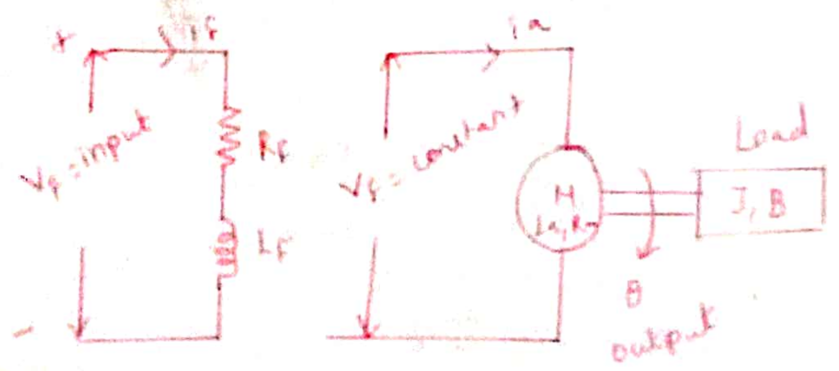
When V_a = const, $T \propto \phi_f$

$I_f \propto \phi$

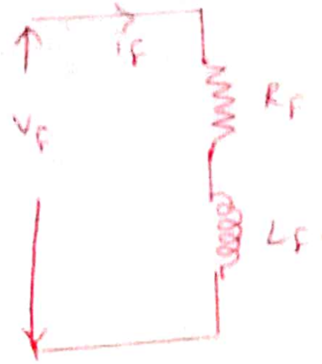
$\therefore T \propto I_f$.

Torque is controlled by controlling the field current.

Reversible operation → Reversing field current



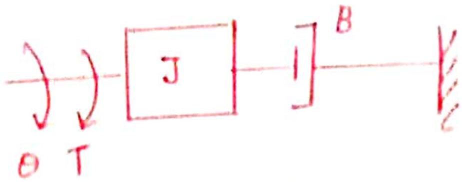
By KVL,



$$V_f = i_f R_f + L_f \frac{di_f}{dt} \rightarrow (1)$$

$T \propto i_f$

$$T = K_{tf}\, i_f \rightarrow (2)$$



$$J \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \rightarrow (3)$$

Taking L.T

$$V_f(s) = R_f I_f(s) + L_f s\, I_f(s) \rightarrow (4)$$

$$T(s) = K_{tf} I_f(s) \rightarrow (5)$$

$$J s^2 \theta(s) + Bs\theta(s) = T(s) \rightarrow (6)$$

equate eqn (6) & (5)

$$(Js^2 + Bs)\,\theta(s) = K_{tf} I_f(s)$$

$$I_f(s) = \frac{(Js^2 + Bs)\,\theta(s)}{K_{tf}} \rightarrow (7)$$

sub (7) in (4)

$$V_f(s) = (R_f + sL_f)\frac{(Js^2+Bs)}{K_{tf}}\theta(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{(R_f + sL_f)(Js^2 + Bs)}$$

Where

$T_f = \frac{L_f}{R_f}$ = field time constant

$T_m = J/B$ = mechanical time constant.

$$\frac{\theta(s)}{V_f(s)} = \frac{k_{tf}}{R_f\left(1 + s\frac{L_f}{R_f}\right) Bs\left(1 + \frac{Js}{B}\right)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{\frac{k_{tf}}{R_f B}}{s(1+sT_f)(1+T_m s)}$$

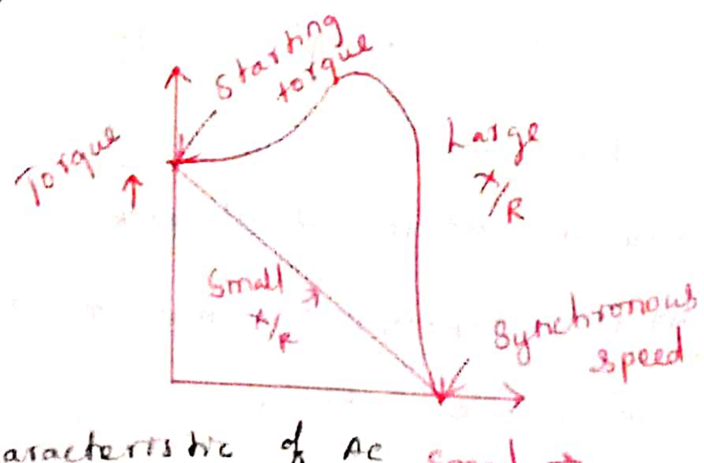
This is the transfer function of field controlled Dc servomotor.

Ac servomotor.

Ac servomotor is basically a two phase induction motor.

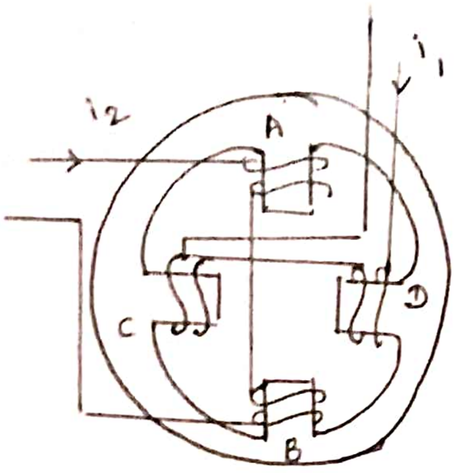
→ Rotor of the servomotor is built with high resistance, X/R ratio is small

→ the excitation voltage applied to two stator winding should have a phase difference of 90°

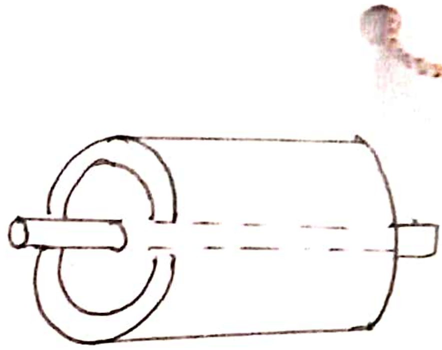


N-T characteristic of Ac servomotor

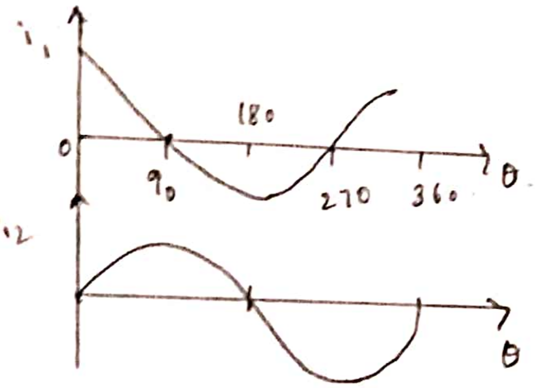
Construction :-



Stator



Rotor .



Exciting currents .

<u>Stator</u> :

→ It consist of two pole pair (AB & CD) mounted on the inner periphery of the stator, such that axes are at an angle of 90° in space

→ Each pole pair carries a winding .

one winding → reference winding

other winding → control winding .

→ Exciting current in the winding should have a phase displacement of 90° .

→ the supply is used to drive the motor is single phase .

Rotor :-

Rotor construction is Squirrel cage or drag cup type

Squirrel cage Rotor :-

→ It is made of Laminations

→ Rotor bars are placed on the slots & SC at both ends by end rings

→ Diameter of the rotor is kept small to reduce inertia & obtain good accelerating characteristics

Drag cup Rotor :-

→ very Low inertia applications

→ Rotor will be in the form of hollow cylinder made of alluminium

→ Aluminium cylinder itself act as short circuited rotor conductors.

Electrically both type of rotors are identical.

working :-

→ The stator windings are excited by voltages of equal rms magnitude and 90° phase difference This results in exciting current i_1 & i_2 that are phase displaced by 90° & have equal rms values

→ These current gives rise to a rotating magnetic field. Which results in clockwise

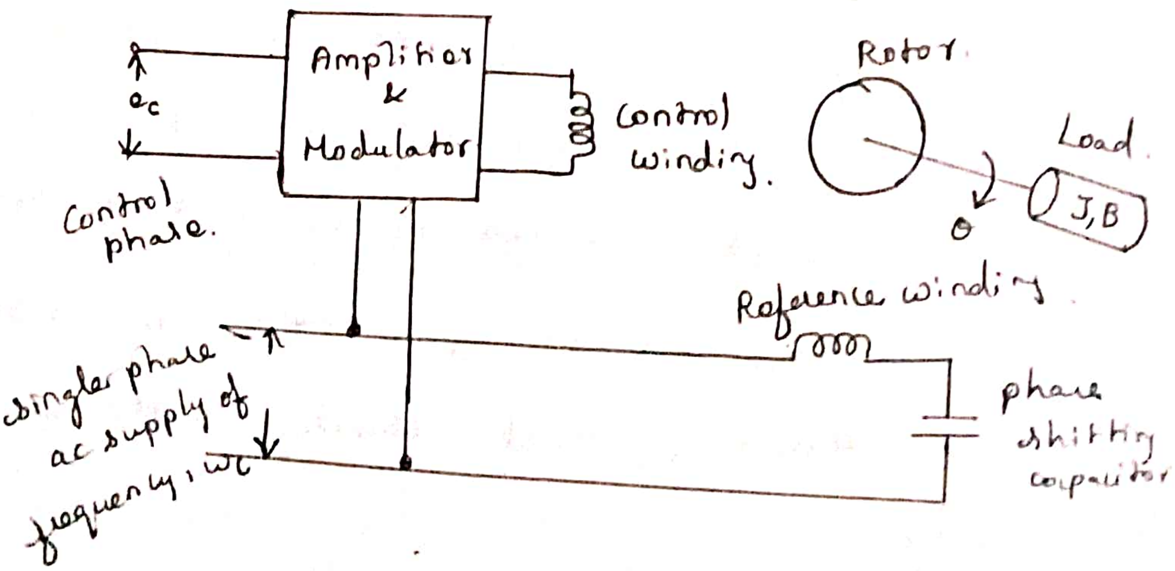
Rotation of the rotor

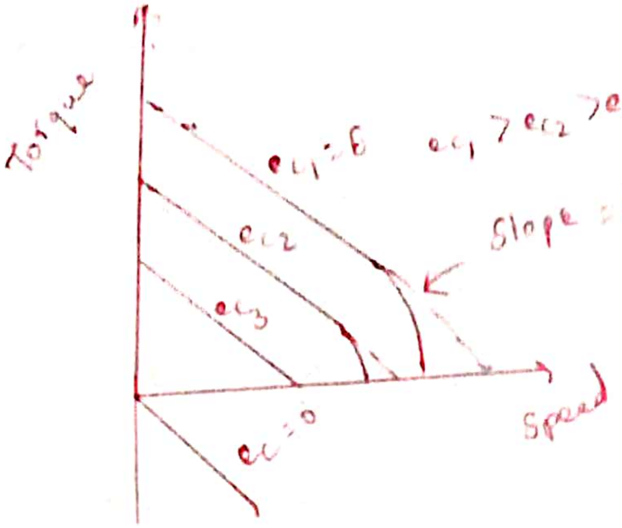
→ The Rotating magnetic field sweeps over the rotor conductor the rotor conductor experiences a change in flux & so voltages are induced in rotor conductor.

→ This voltage circulates current in the short circuited rotor conductors & the current create rotor flux.

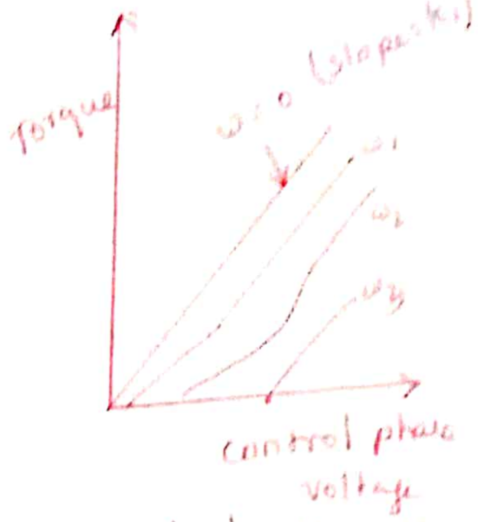
→ Due to this interaction of stator & rotor flux, a mechanical force (torque) is developed on the rotor & so the rotor starts moving in the same direction as that of rotating magnetic field.

→ The reference winding is excited by a constant voltage source & the control winding is excited by the modulated control signal & this voltage is of variable magnitude & polarity.





Speed torque curve of an ac servomotor.



control voltage Vs Torque curves

For constant speed, except near zero speed the torque does not vary linearly with respect to input voltage e_c.

Transfer function of Ac servomotor

Let, T_m - Torque developed by servomotor

θ - angular displacement of rotor

$\omega = \frac{d\theta}{dt}$ = Angular speed

T_l - Torque required by Load.

J - moment of inertia of Load and rotor

B - viscous frictional coefficient of Load & rotor

$K_1 \rightarrow$ Slope of control phase voltage Vs Torque

$K_2 \rightarrow$ Slope of Speed - Torque characteristic under Assumption,

Torque developed by the motor is represented by

$$T_m = k_1 e_c - k_2 \frac{d\theta}{dt} \rightarrow (1)$$

Rotating part of the motor & the Load can be modelled by

Load torque $T_l = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \rightarrow (2)$.

At equilibrium

Motor torque = Load torque

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = k_1 e_c - k_2\frac{d\theta}{dt}.$$

Taking Laplace transform.

$$Js^2\theta(s) + BS\theta(s) = k_1 E_c(s) - k_2 S\theta(s).$$

$$\left[Js^2 + BS + k_2 S\right]\theta(s) = k_1 E_c(s)$$

$$\frac{\theta(s)}{E_c(s)} = \frac{k_1}{S\left[JS + (B + k_2)\right]}.$$

$$= \frac{\frac{k_1}{(B + k_2)}}{S\left[\frac{J}{B+k_2}S + 1\right]}$$

Where,

$$k_m = \frac{k_1}{B + k_2} = \text{motor gain constant}$$

$$\tau_m = \frac{J}{B + k_2} = \text{motor time constant.}$$

$$\boxed{\frac{\theta(s)}{E_c(s)} = \frac{k_m}{s(\tau_m s + 1)}}$$

This is Transfer function of ac servomotor.

Block diagram :-

A block diagram is the pictorial representation of the functions performed by each component & of the flow of signals.

Such diagram depicts the interrelationships that exist among the various components.

The element of block diagram are

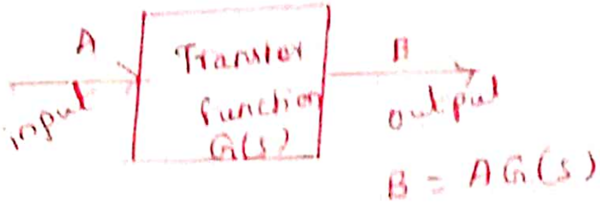
→ Block

→ branch point

→ Summing point.

Block :-

Functional block or block is a symbol for the mathematical operation on the input signal to the block that produce the output.

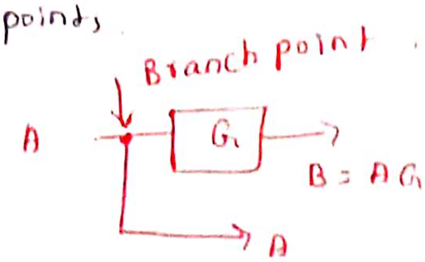


Summing point :-

It is used to add two or more signals in the system.

Branch point :-

It is a point from which the signal from the block goes concurrently to other blocks or summing points.



1. Construct the block diagram of armature controlled dc motor

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \rightarrow (1)$$

$$T = k_t i_a \rightarrow (2)$$

$$T = J\frac{d\omega}{dt} + B\omega \rightarrow (3)$$

$$E_b = k_b \omega \rightarrow (4)$$

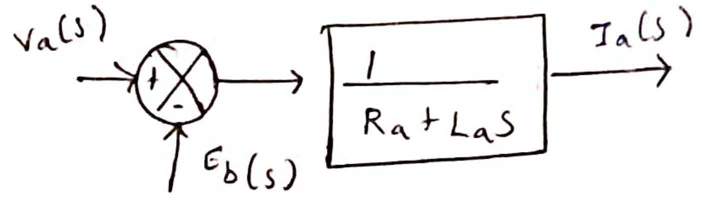
$$\omega = \frac{d\theta}{dt} \rightarrow (5)$$

Taking Laplace Transform

(1) $\Rightarrow$ $V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$

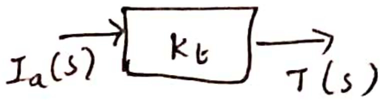
$$V_a(s) - E_b(s) = I_a(s)[R_a + L_a s]$$

$$I_a(s) = \frac{V_a(s) - E_b(s)}{(R_a + L_a s)}$$



(2) $\Rightarrow$ $T = k_t i_a$

$$T(s) = k_t I_a(s)$$

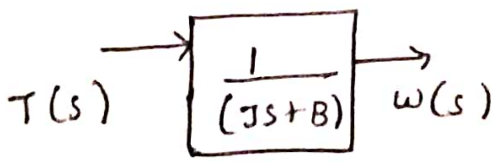


(3) $\Rightarrow$ $T = J\frac{d\omega}{dt} + B\omega$

$$T(s) = Js\,\omega(s) + B\omega(s)$$

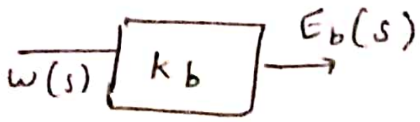
$$T(s) = (Js + B)\omega(s)$$

$$\omega(s) = \frac{1}{(JS + B)} T(s)$$



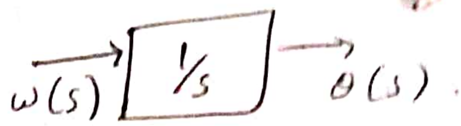
(4) $\Rightarrow$ $E_b = k_b \omega$

$$E_b(s) = k_b \omega(s)$$

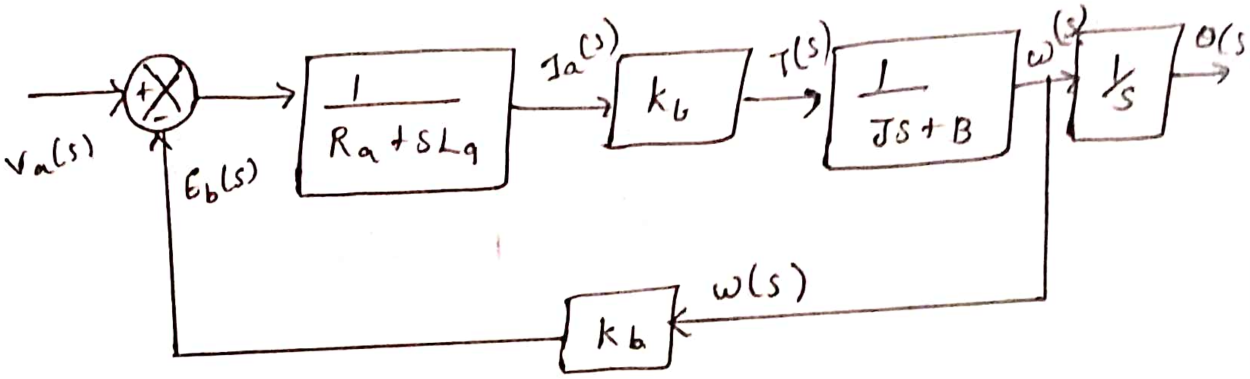


(n) $\Rightarrow$ $\omega = \dfrac{d\theta}{dt}$

$\omega(s) = s\theta(s)$



$\theta(s) = \dfrac{1}{s}\omega(s)$

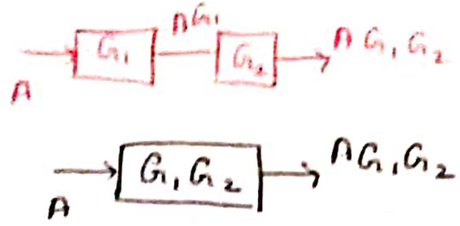


Block diagram Reduction :-

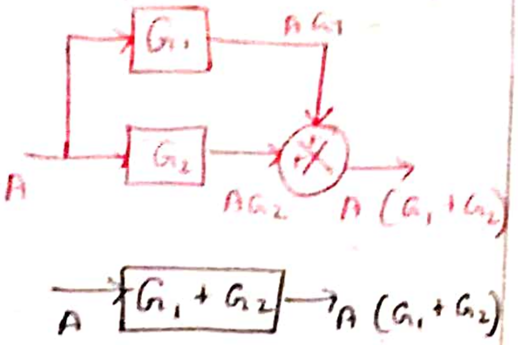
The block diagram can be reduced to find the overall transfer function of the system.

The rules are framed such that any modification made on the diagram doesn't alter the input & output relation.

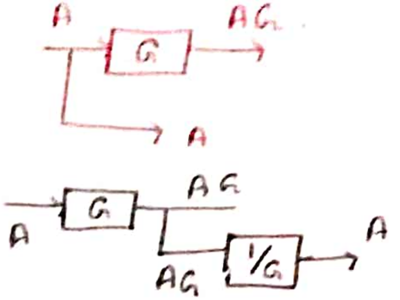
Rule 1: Combining the blocks in cascade.



Rule 2: Combining parallel blocks (or) Combining feed forward paths

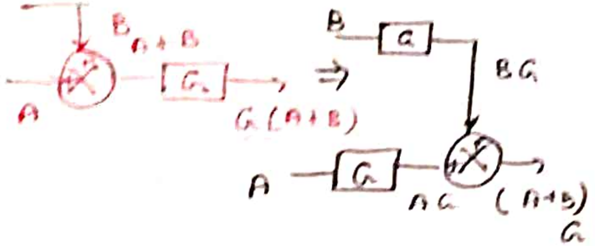


Rule 3: Moving the branch point ahead of the block

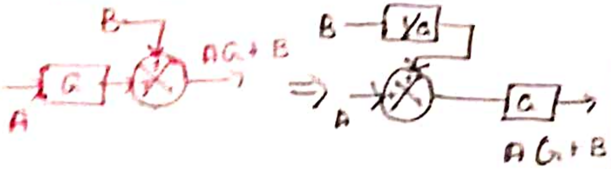


Rule 4: moving branch point before the block

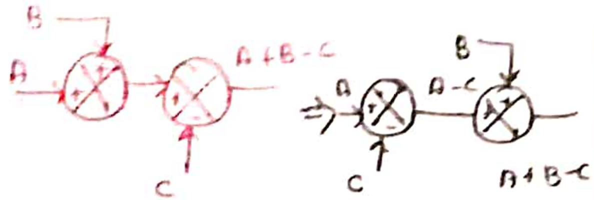
Rule 5:- moving the summing point ahead of the block



Rule 6: moving summing point before the block

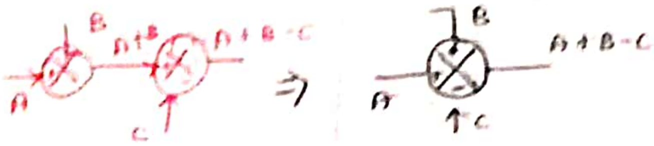


Rule 7: Interchanging summing point

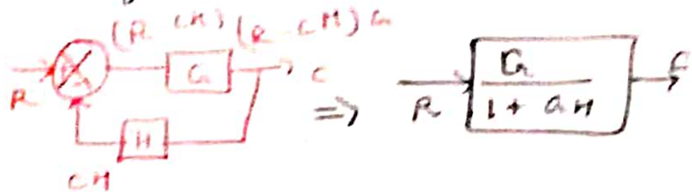


Rule 8:- Splitting summing point

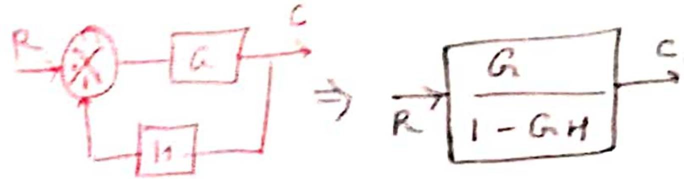
Rule 9:- Combining summing point



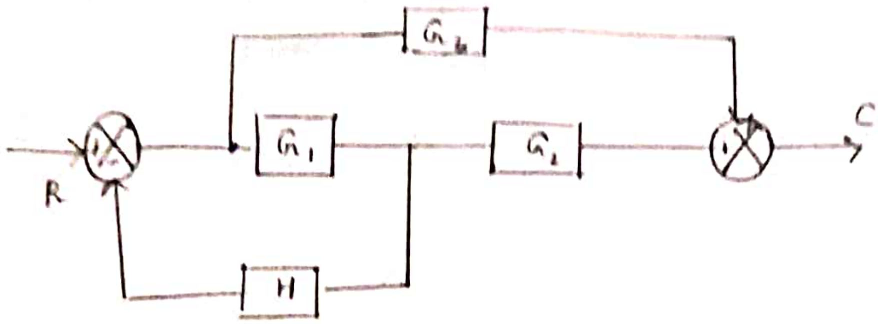
Rule 10:- elimination of (negative) feedback loop



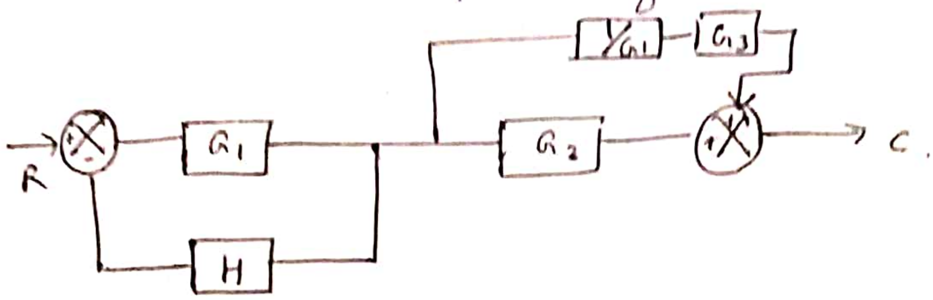
Rule 11:- elimination of (positive) feedback loop



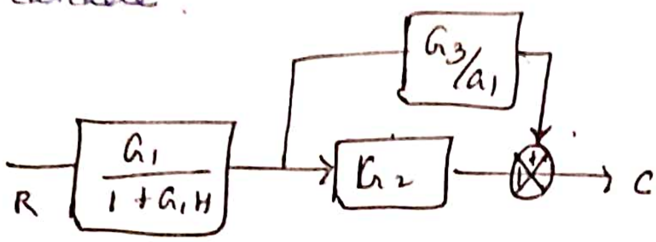
Reduce the block diagram shown in figure & find C/R



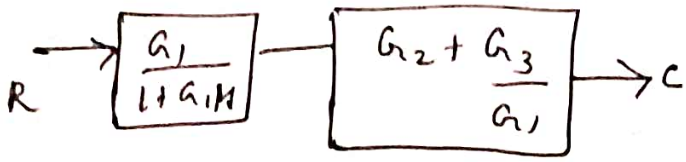
Step 1: move the branch point after the block G_1



Step 2: Eliminate the feedback path & combining blocks in cascade.



Step 3: Combining block in cascade.



$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right)$$

$$\boxed{C/R = \frac{G_1G_2 + G_3}{1+G_1H}}$$

Signal flow graph -

It is used to represent the control System graphically It was developed by S J Mason

A signal flow graph is a diagram that represents a set of simultaneous Linear algebraic equations

Advantage

→ Simpler

→ over all gain of the system can be computed easily

→ A signal flow graph consist of a network in which nodes are connected by directed branches.

→ Each node represents a system Variable & each branch connected between two nodes act as a single multiplier.

→ Each branch has a gain or transmittance when the signal pass through a branch, it gets multiplied by the gain of the branch

→ Signal flow in only one direction.

→ the direction of signal flow is indicated by an arrow mark.

Node → It is a point representing a variable (or) signal.

Branch → A branch is directed line segment joining two nodes

Transmittance → The gain acquired by the signal when it travels from one node to another.

→ It can be real or complex

Input node (Source → It has only outgoing branches.

output node (sink) → It has only incoming branches

path → It is a traversal of connected branches in the direction of the branch arrows. the path should not cross a node more than once.

Open path → It starts at a node & ends at another node

closed path → It starts & end at same node.

Forward path → It is a path from input to output node that doesn't cross any node more than once.

forward path gain → It is the product of the branch transmittance (gain) of a forward path.

Individual Loop :- It is a closed path starting (2n)

from a node & after passing through a certain part of a graph arrives at same node without crossing any node more than once.

Loop gain :- It is the product of the branch transmittance (gains) of a Loop

Non-touching Loop :- If the Loop doesn't have a common node, then they are said to be non-touching Loop.

properties of signal flow graph :-

→ It is applicable to Linear system.

→ Algebric equation are used

→ System is not unique.

→ node represents variable or signal.

Signal flow Reduction

Signal flow graph of a system can be reduced either by using the rules of a signal flow algebra (or) by using Mason's gain Formula.

Mason's gain formula:

It is used to determine the transfer function of the system from the signal flow graph of the system.

Let,
$R(s)$ - input to the system

$C(s)$ - output of the system

Transfer function $T(s) = \frac{C(s)}{R(s)}$

Mason's gain formula states the overall gain of the system as follows.

$$\text{overall gain } T = \frac{1}{\Delta} \sum_{k} P_k \Delta_k$$

Where,

$T \rightarrow$ Transfer function

P_k - forward path gain of k^{th} forward path.

k - Number of forward path in the signal flow graph

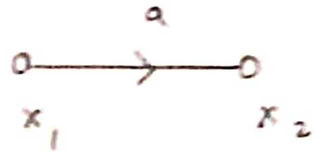
Δ = 1 - (Sum of individual loop gain) + (Sum of gain product of all possible combination of two non-touching loops) - (sum of gain product of all possible combination of three non touching loop + ...

Δ_K - Δ for that part of graph which is not touching k^{th} forward path.

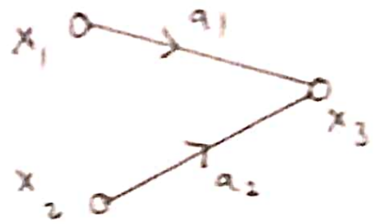
Signal flow graph Algebra :

It can be reduced to obtain the Transfer function of the system using following rules

Rule 1 :-



$x_2 = x_1 a$

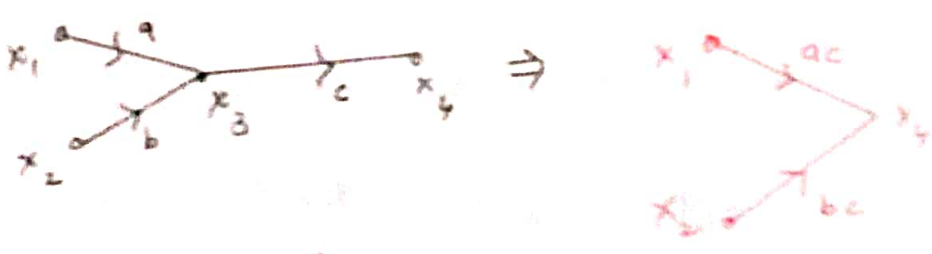


$x_3 = a_1 x_1 + a_2 x_2$

Rule 2 :

Rule 3 :

Rule 4 :-



Rule 5 :

1. Construct a signal flow graph for armature controlled dc motor.

Solution :-

Step 1: The differential equation governing the armature controlled dc motor.

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b$$

$$T = K_t i_a$$

$$T = J \frac{d\omega}{dt} + B\omega$$

$$e_b = K_b \omega$$

$$\omega = \frac{d\theta}{dt}$$

Step 2 :- Taking Laplace Transform

$$V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$$

$$T(s) = K_t I_a(s)$$

$$T(s) = Js\,\omega(s) + B\omega(s)$$

$$E_b(s) = K_b \omega(s)$$

$$\omega(s) = s\,\theta(s)$$

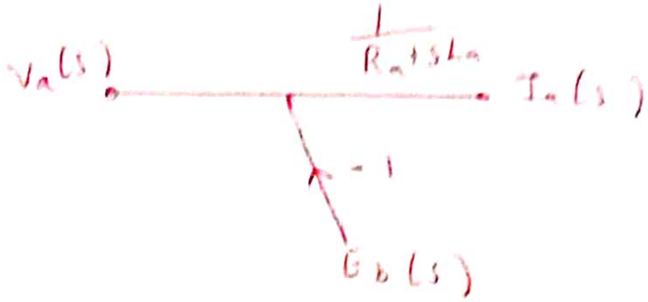
Step 3 :- Signal flow graph.

$$V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$$

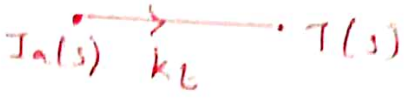
$$V_a(s) - E_b(s) = (R_a + L_a s) I_a(s)$$

$$I_a(s) = \frac{V_a(s) - E_b(s)}{R_a + L_a s}$$

Loop ...



$$T(s) = k_t I_a(s)$$



$$T(s) = (Js+B)\,\omega(s)$$

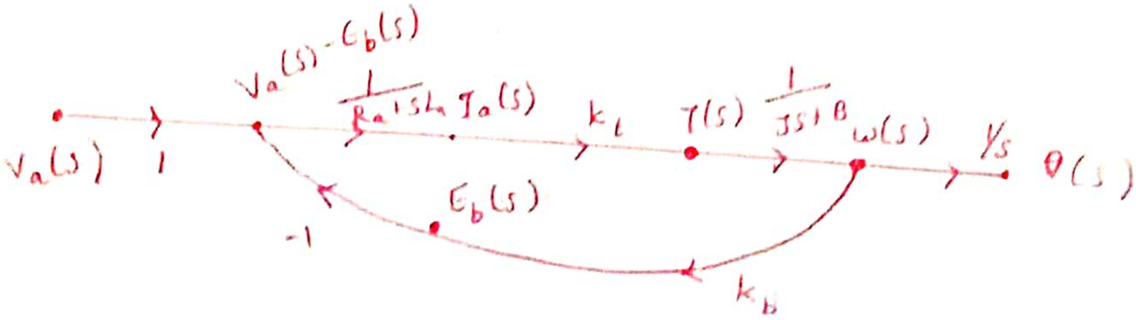
$$\omega(s) = \frac{T(s)}{Js+B}$$

$$E_b(s) = k_b\,\omega(s)$$

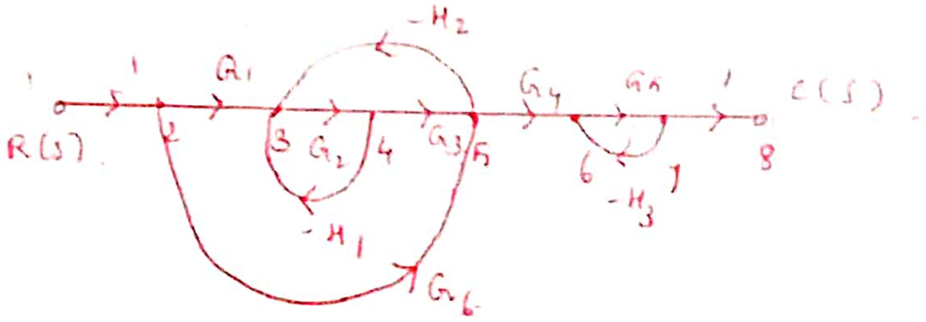
$$\omega(s) = s\,\theta(s)$$

$$\theta(s) = \frac{1}{s}\,\omega(s)$$

Step 4 :- overall signal flow graph.



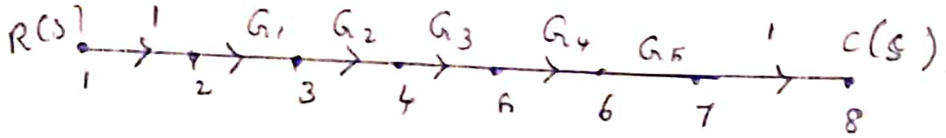
1) To find the overall transfer function of the system whose signal flow graph.



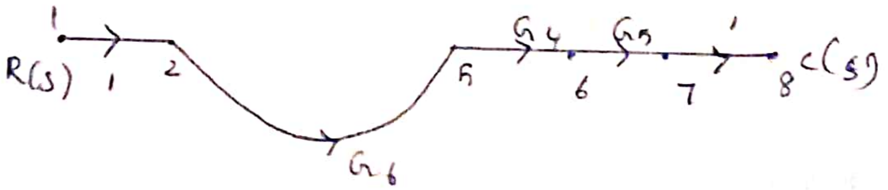
Solution:-

Step 1) Forward path gain.

$k = 2$

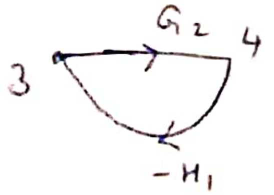


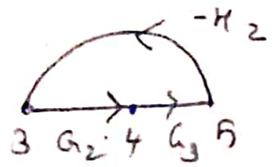
$P_1 = G_1 G_2 G_3 G_4 G_5$

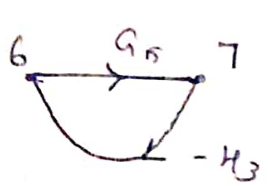


gain $P_2 = G_6 G_4 G_5$

Step 2:- Individual Loop gain







Loop 1 Loop 2 Loop 3.

$L_1 = -G_2 H_1$ $L_2 = -G_2 G_3 H_2$ $L_3 = -H_3 G_5$

Loop

Step 3 :- Gain product of two non touching loops

$$P_{12} = G_2 H_1 G_5 H_3$$

$$P_{22} = G_2 G_3 H_2 G_5 H_3$$

Step 4: Transfer function

By Mason's Gain formula

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k \qquad k = 2$$

$$= \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} \rightarrow (1)$$

$$\Delta_1 = 1 - 0 = 1$$

$$\Delta_2 = 1 - L_1 = 1 + G_2 H_1$$

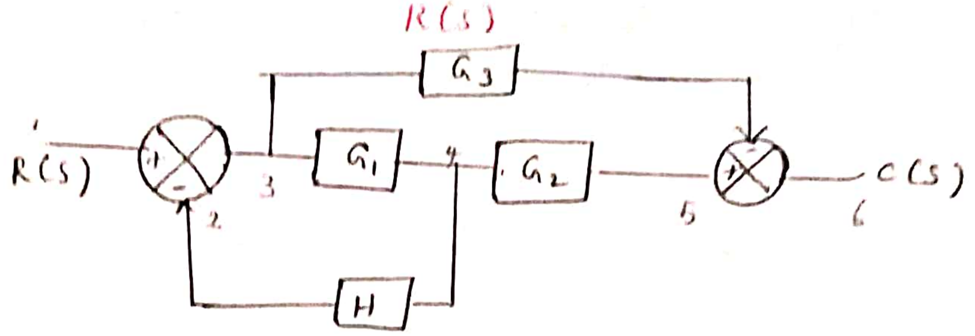
$$\Delta = 1 - (L_1 + L_2 + L_3) + P_{12} + P_{22}$$

$$= 1 - (-G_2 H_1 - G_2 G_3 H_2 - H_3 G_5) + G_2 H_1 G_5 H_3 + G_2 G_3 G_5 H_2 H_3$$

$$\Delta = 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_{12} G_3 G_5 H_2 H_3$$

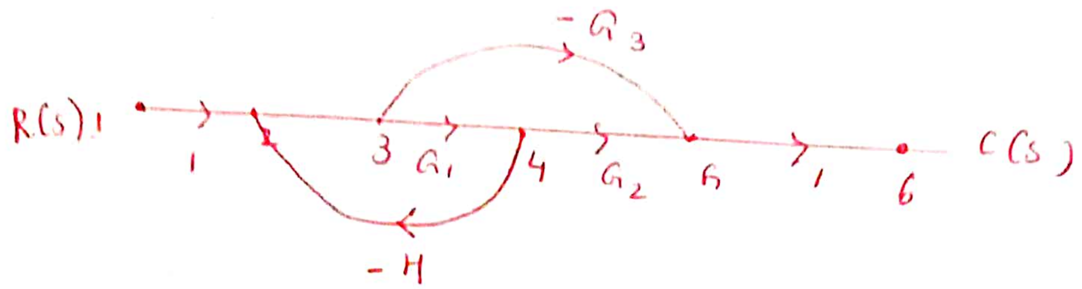
$$(1) \Rightarrow T = \frac{G_1 G_2 G_3 G_4 G_5 + G_6 G_4 G_5 (1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

2) Convert the given block diagram to signal flow graph & determine $C(s)$



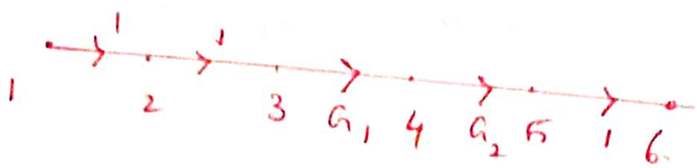
Solution:

Step 1: Signal flow diagram.

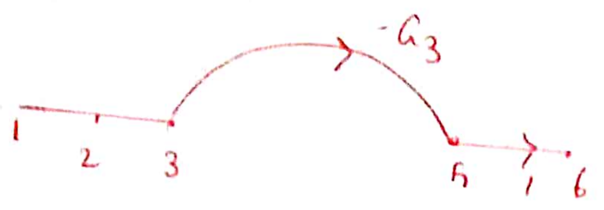


Step 2: Forward path Gain

$k = 2.$



$P_1 = G_1 G_2$



$P_2 = -G_3$

Loop

Step 3 : Individual Loop gain.

$L_1 = -G_1 H$

Step 4 : Gain product of two Non touching Loop.

No combination of Loop.

Step 5 : Transfer function.

By Mason's gain formula

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k \qquad k = 2$$

$$= \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$\Delta_1 = 1 - 0 = 1$

$\Delta_2 = 1 - 0 = 1$

$\Delta = 1 - (L_1) + 0$

$= 1 + G_1 H$

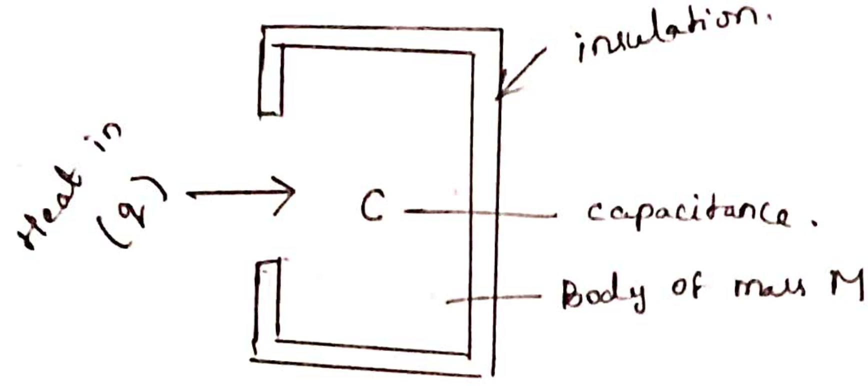
$$\boxed{T = \frac{G_1 G_2 - G_3}{1 + G_1 H}}$$

Thermal Systems → Transfer of heat from one substance to another

Items	Electrical S/m	Thermal S/m.
input	charge (q)	heat flow rate (Q) (H)
output	voltage (V)	temperature ($\Delta\theta$)
Elements.	Resistance $R = \frac{V}{q(i)}$	Thermal resistance. $R = \frac{\Delta\theta}{Q\ H}$ $R = \frac{1}{Q\,Cs}$
	Capacitance $q = C\frac{dv}{dt}$ Where, C = Capacitance. V → voltage q → charge.	thermal capacitance $Q = C\frac{d\Delta\theta}{dt}$ Where, $\frac{d(\Delta\theta)}{dt}$ → rate of change of Temp. Q → heat flow $C = MS$ M → mass of the body. (gm) S → Specific heat of the material (cal/gm.

consider an example,

A heat flowing into a body with thermal capacitance C causes a temperature T to rise above the ambient value T_0.



Heat transmission in body is classified into

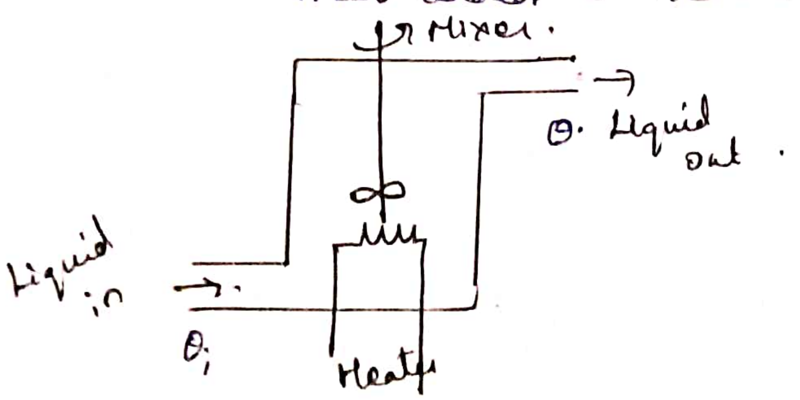
1) conduction → transmission through body.
2) convection → transmission & mixing.
3) Radiation → electromagnetic waves.

The rate of flow of heat through a body is ~~called~~ determined by thermal Resistance.

$$R_T = \frac{T_2 - T_1}{q}.$$

At equilibrium.

Heat added = Heat stored + heat removed.



Assumption,

1) Fluid in the tank is perfectly mixed. so that temp = const

2) Tank is insulated → to eliminate heat Loss.

∴ No heat storage in the insulation.

Let

θ_i → steady state temperature of inflowing Liquid.

θ → steady state temperature of outflowing Liquid

ΔH → heat rate inflow ~~flow inside the body~~

ΔH_1 → heat rate outflow ~~flow outside the body~~.

ΔH_2 → heat storage rate

$\boxed{\begin{matrix} Q \Rightarrow H \\ T \to \Delta\theta \end{matrix}}$

$$\Delta H = \Delta H_1 + \Delta H_2$$

$$\Delta H_1 = \frac{T}{R} = \frac{\Delta\theta}{R}.$$

$$\Delta H_2 = C\frac{d\Delta\theta}{dt}$$

$$= (M c_s)\frac{d\Delta\theta}{dt}.$$

thermal capacitance.

$$\therefore \Delta H = \Delta H_1 + \Delta H_2$$

$$= \frac{\Delta\theta}{R} + C\frac{d\Delta\theta}{dt}.$$

Taking L.T

$$\Delta H(s) = \frac{\Delta\theta(s)}{R} + Cs\,\Delta\theta(s).$$

$$= \left[\frac{1 + RCs}{R} \right] \Delta\theta(s)$$

$$\frac{\Delta H(s)}{\Delta\theta(s)} = \frac{1 + RCs}{R}$$

$$= \frac{R}{1 + RCs}$$

$$\frac{\Delta\theta(s)}{\Delta H(s)} = \cancel{1+RCs}\ \frac{R}{1 + RCs}$$

16)

18) $\rightarrow$

21) $\rightarrow$ MC $\rightarrow$ 2nd 3rd

UNIT - II

TIME RESPONSE

Time Response :-

The time response of the system is the output of the closed Loop system as a function of time. It is denoted by $c(t)$.

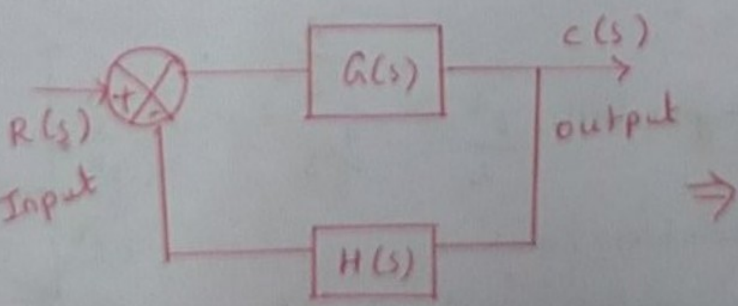
The time response of a control system consist of two parts.

1) Transient response
2) Steady state response.

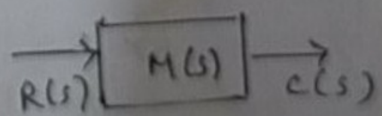
Transient response → It is the response of the system when the input changes from one state to another.

Steady state Response → It is the response as time, t approaches infinity.

Closed Loop Transfer function.



⇒



$$M(s) = \frac{G(s)}{1+G(s)H(s)}$$

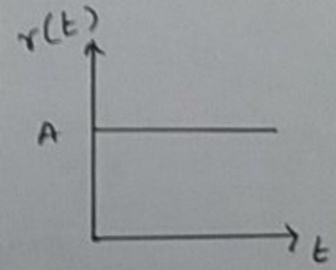
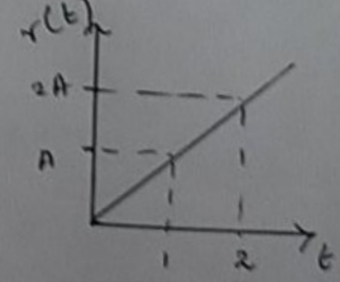
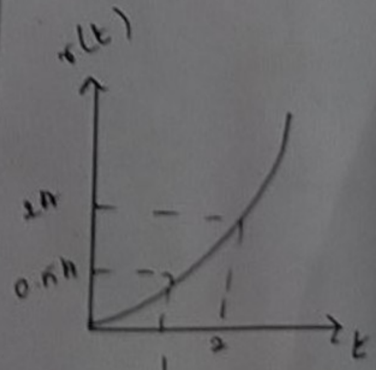
closed Loop

$$T.F.\ \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Test signal → Simple functions for time.

The standard test signals are.

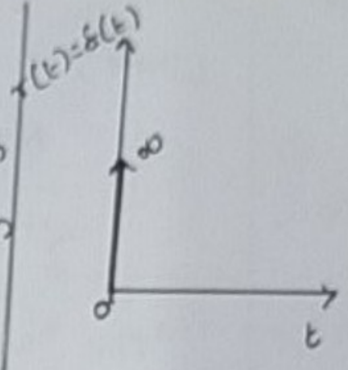
1. Step signal
2. Ramp signal
3. parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Name of signal.	Time domain equation of signal $r(t)$	Laplace T/f. $R(s)$	Graph.
1.a) Step signal Whose values changes from (0 to A) at $t=0$ and remain constant at A for $t>0$	A unit step: 1 Resembles: Actual steady input to system	A/s $1/s$	
2) Ramp signal. Whose values Linearly increases with time from initial value of zero, at $t=0$.	At unit ramp: t Resembles: constant velocity input to system.	A/s^2 $1/s^2$	
3) parabolic signal Whose values Varies as square of the time from initial value of zero at $t=0$.	$\frac{At^2}{2}$ unit parabolic: $\frac{t^2}{2}$ Resembles: Constant acceleration input to the system	$\frac{A}{s^3}$ $1/s^3$	

4.	Impulse signal	$\delta(t)$	1

(A signal of very Large magnitude which is available for very short duration is called impulse signal.

unit impulse $R(s) = 1$



The response of the system with input as impulse signal is called (weighting function) of the System.
(or)
Impulse response.

Order of the System:-

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called n^{th} order system.

Transfer function $T(s) = \dfrac{P(s)}{Q(s)} = \dfrac{b_0 s^m + b_1 s^{m-1} \dots + b_m}{a_0 s^n + a_1 s^{n-1} \dots + a_n}$

Where,

$P(s) \rightarrow$ Numerator polynomial

$Q(s) \rightarrow$ Denominator polynomial.

The order of the system is given by the Maximum power of s in the denominator polynomial $Q(s)$

$$Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} \dots + a_n$$

$n \rightarrow$ order of the system

When, $n=0$, the system is zero order system

$n=1$, the system is 1st order system

$n=2$, the system is 2nd order system

& so on

The order can be specified for both open Loop system & closed Loop system.

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

Where,

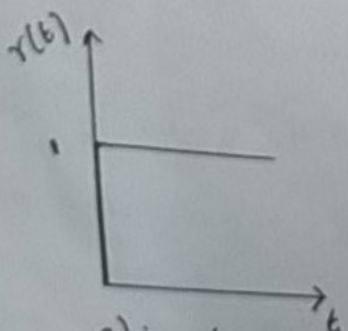
$z_1, z_2 \ldots z_m \rightarrow$ zeros of the system.

$P_1, P_2 \ldots P_m \rightarrow$ poles of the System.

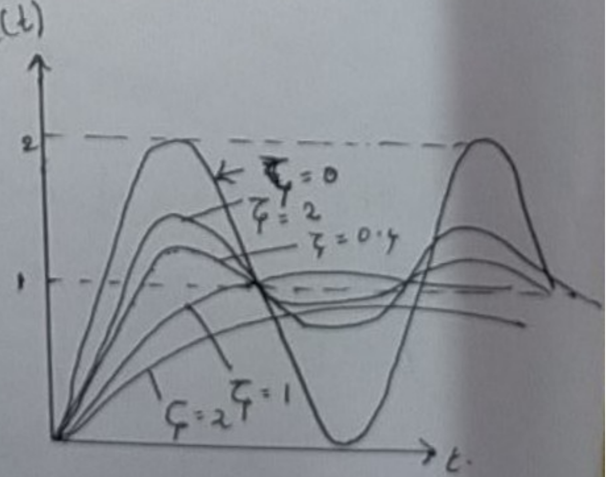
Time domain specification :-

The desired performance characteristics of control systems are specified in terms of time domain specification.

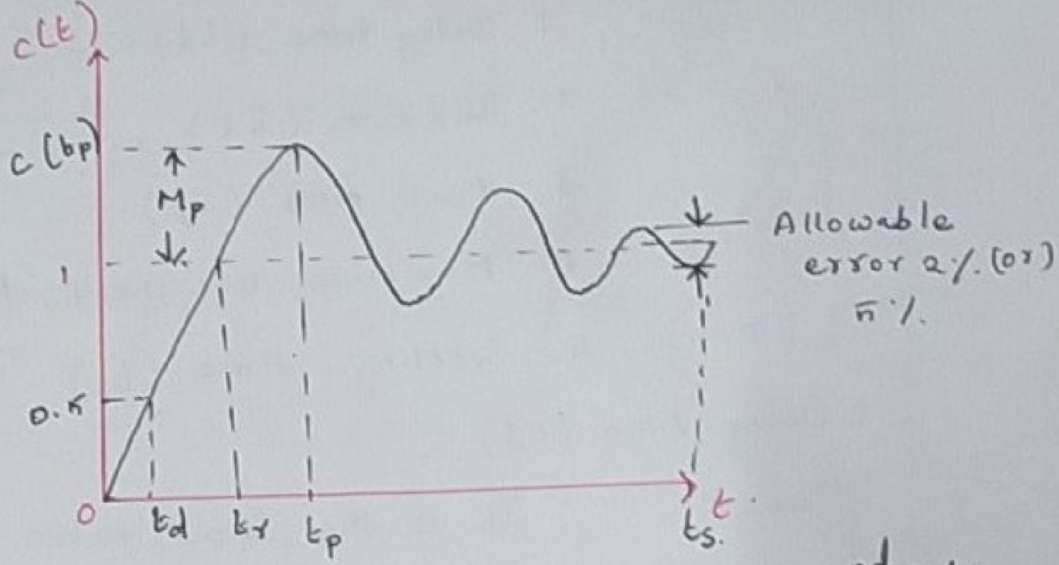
Consider a unit step input signal in terms of transient response.



a) input



Response.



Damped Oscillatory Response of 2nd order system for unit step input. Step

The transient response of a system to a unit step input depends on the initial conditions.

practical standard:-

The system at rest, output and all time derivatives before t=0 will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state.

The transient response characteristics of a control system to a unit step input specified in terms of the following time domain specification.

1. Delay time, (t_d)

2. Rise time (t_r)

3. Peak time (t_p)

4. Maximum overshoot, M_p

5. Settling time (t_s)

1. Delay time (t_d):

It is the time taken for response to reach 50% of the final value, for every first time.

2. Rise time (t_r): It is the time taken for response to raise from 0 to 100% for the very first time.

for

under damped → t_r → 0 - 100%

over damped → t_r → 10% - 90%

critical damped → t_r → 5% - 95%

Expression for time domain specification.

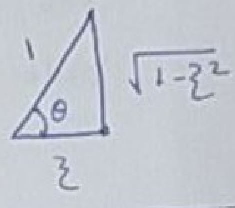
the unit step response of 2nd order system for under damped is given by

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \rightarrow (1)$$

When $t = t_r$, $c(t) = 1$

$$1 - \frac{e^{-\zeta \omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

$$-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r+\theta)=0$$



$$\sin(\omega_d t_r+\theta)=0 \quad , e^{-\zeta\omega_n t_r}\neq 0$$

$$\tan\theta=\frac{\sqrt{1-\zeta^2}}{\zeta}$$

When $\varphi=0,\pi,2\pi\ldots, \sin\varphi=0$

$$\theta=\tan^{-1}\sqrt{\frac{1-\zeta^2}{\zeta}}$$

$$\omega_d t_r+\theta=\pi$$

$$\omega_d t_r=\pi-\theta$$

$$\boxed{\text{Rise time} \quad t_r=\frac{\pi-\theta}{\omega_d}}$$

damped frequency of oscillation $\omega_d=\omega_n\sqrt{1-\zeta^2}$

$$\therefore\ t_r=\frac{\pi-\tan^{-1}\sqrt{\frac{1-\zeta^2}{\zeta}}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec.}$$

3. Peak time (t_p): It is the time taken for the response to reach the peak value the very first time.

(or)

It is the time taken for the response to reach the peak overshoot, M_p.

Expression for time domain specification.

The unit step response of under damped second order system

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta).$$

For peak time t_p differentiate $c(t)$. & equate to zero

$$\left.\frac{d}{dt} c(t)\right|_{t=t_p} = 0.$$

$$\frac{d}{dt} c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (-\zeta\omega_n) \sin(\omega_d t + \theta) + \left[-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right] \cos(\omega_d t + \theta)\omega_d$$

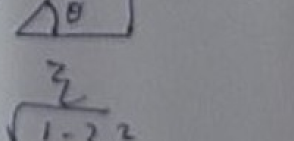
$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \zeta\omega_n \sin(\omega_d t + \theta) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t + \theta)\omega_d$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \zeta\omega_n \sin(\omega_d t + \theta) - \frac{\omega_n \cancel{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}}{\cancel{\sqrt{1-\zeta^2}}} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2} \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right)$$



$$\sin\theta = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sin(\omega_d t + \theta) - \theta\right]$$

$$\cos\theta = \zeta$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t).$$

At $t = t_p$, $\frac{d}{dt} c(t) = 0$.

$$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin \omega_d t_p = 0$$

$e^{-\zeta\omega_n t_p} \neq 0$, $\sin \omega_d t_p = 0$.

$\varphi = 0, \pi, 2\pi \ldots$ $\sin\varphi = 0$.

$$\omega_d t_p = \pi$$

$$\boxed{\text{peak time. } t_p = \frac{\pi}{\omega_d}}$$

the damped frequency of oscillation $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}.$$

4. peak overshoot (M_p)

It is defined as the ratio of Maximum peak value to the final value. where maximum peak value is measured from final value.

$c(\infty)$ = final value of $c(t)$.

$c(t_p)$ - maximum value of $c(t)$.

peak overshoot, $M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$

% peak overshoot, $\% M_p = \dfrac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

The unit step response of 2nd order system is given by

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta).$$

At $t = \infty$

$$c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d(\infty) + \theta\right]$$

$$= 1 - 0 = 1$$

At $t = t_p$

$$c(t_p) = 1 - \frac{e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta).$$

Where $t_p = \dfrac{\pi}{\omega_d}$.

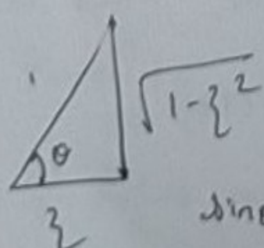
$$= 1 - \frac{e^{-\zeta \omega_n \cdot \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\cancel{\omega_d} \cdot \frac{\pi}{\cancel{\omega_d}} + \theta\right)$$

$$= 1 - \frac{e^{-\zeta \omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin(\pi + \theta).$$

Where, $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$\boxed{\sin(\pi + \theta) = -\sin\theta}$

$$= 1 + \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\theta$$



$\sin\theta = \sqrt{1-\zeta^2}$

$$= 1 + \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2}$$

$$= 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

% peak overshoot, $\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

$$= \frac{1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$\boxed{\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100}$$

5) Settling time (T_s)

It is defined as the time taken by the response to reach & stay within a specified error. It is usually expressed as % of final value.

Tolerable error → 2% or 5% of final value.

The response of 2nd order system has two components. They are

1. Decaying exponential component $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$
2. Sinusoidal component $\sin(\omega_d t + \theta)$

The decaying exponential term dampens (or) reduces the oscillation produced by sinusoidal component. Hence settling time is decided by exponential component.

For 2% tolerance error band, $t = t_s$

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$$

$e^{-\zeta\omega_n t_s} = 0.02$, since ζ = least value.

Taking Logarithm on both sides.

$$-\zeta\omega_n t_s = \ln 0.02$$

$$-\zeta\omega_n t_s = -4$$

$$t_s = \frac{4}{\omega_n \zeta}$$

For 2nd order system, time constant $T = \frac{1}{\zeta\omega_n}$.

$\therefore$ settling time $\boxed{t_s = \frac{4}{\zeta\omega_n} = 4T}$ (for 2% error)

for 5% error.

$$e^{-\zeta\omega_n t_s} = 0.05$$

$$-\zeta\omega_n t_s = \ln 0.05$$

$$-\zeta\omega_n t_s = -3$$

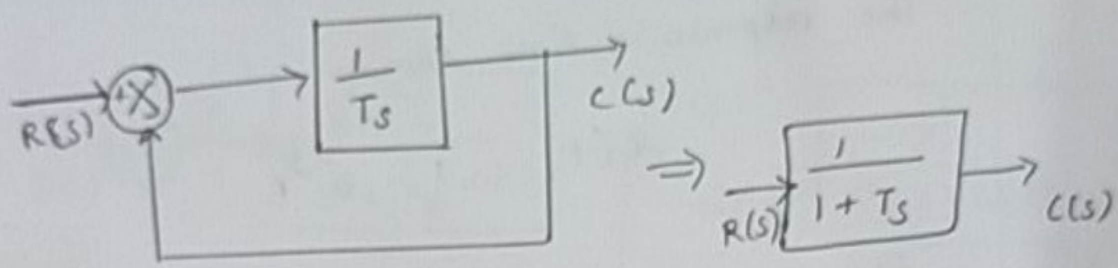
$$t_s = \frac{3}{\zeta\omega_n}$$

$\boxed{t_s = 3T}$ for 5% error

% settling time $t_s = \frac{\ln(\%\ \text{error})}{\zeta\omega_n} = \ln(\%\ \text{error})\,T$

Response of 1st order system for unit step input

The closed loop order system with unity feedback.



Transfer function $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

unit step input $R(s) = \frac{1}{s}$.

$$C(s) = \frac{1}{s(1+Ts)}.$$

$$= \frac{1}{Ts\left(s+\frac{1}{T}\right)} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}.$$

Taking partial fraction.

$$C(s) = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{T}\right)}$$

$$\frac{1}{T} = A\left(s+\frac{1}{T}\right) + Bs \quad \rightarrow (1)$$

put $s=0$ $\quad \frac{1}{T} = A\frac{1}{T}$

$$\boxed{A=1}$$

$s = -\frac{1}{T}$ $\quad \frac{1}{T} = B\left(-\frac{1}{T}\right)$

$$B = -1$$

$$c(s) = \frac{1}{s} - \frac{1}{\left(s + \frac{1}{T}\right)}$$

The response in time domain

$$c(t) = 1 - e^{-t/T}$$

For closed Loop 1st order system.

unit step response $= 1 - e^{-t/T}$

Step response $= A\left(1 - e^{-t/T}\right)$

When $t = 0$, $c(t) = 1 - e^{0} = 0$

$t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

$t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

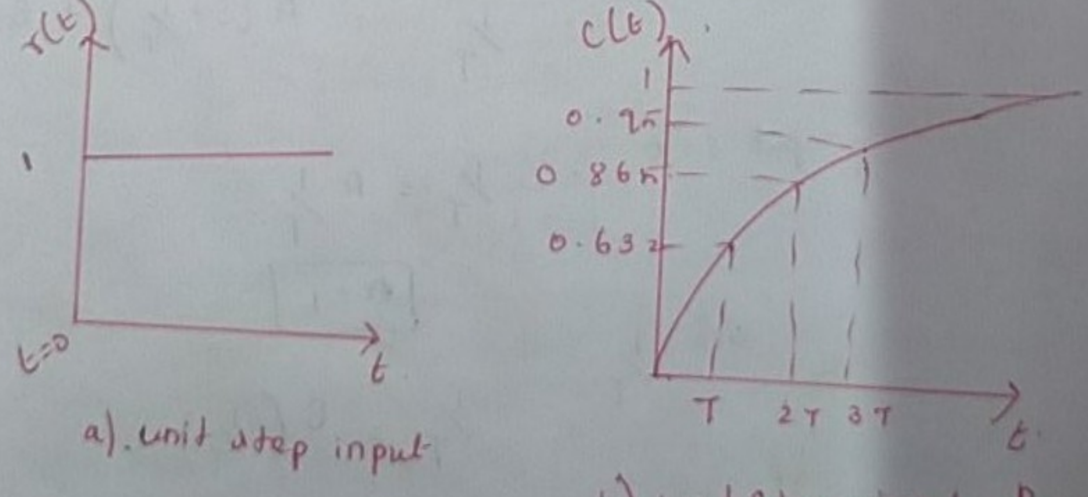
$t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

$t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$

$t = 5T$, $c(t) = 1 - e^{-5} = 0.993$

$t = \infty$, $c(t) = 1 - e^{-\infty} = 1$

$T \rightarrow$ time constant of the system.

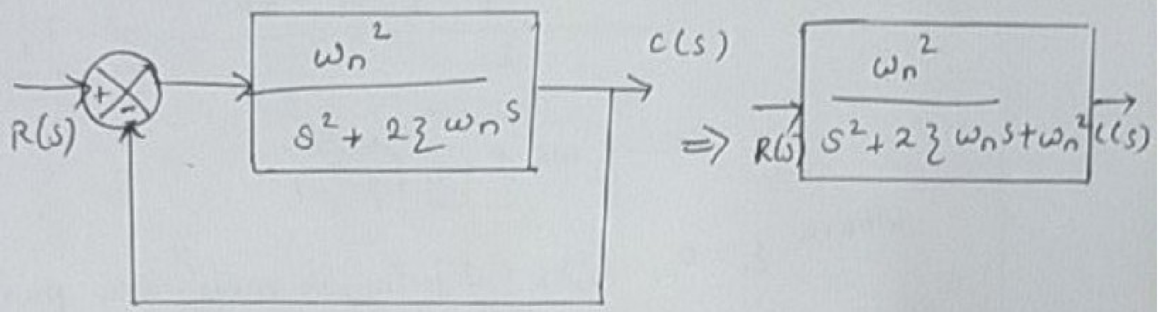


a). unit step input

b) unit step input Response.

2nd order system.

The closed Loop second order system.



Transfer function $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$.

Where, $\omega_n \rightarrow$ undamped natural frequency, rad/sec.

$\zeta \rightarrow$ Damping ratio.

Damping ratio:- It is defined as the ratio of the actual damping to the critical damping.

The responses $c(t)$ of second order system depends on the values of damping ratio.

Depending upon ζ, the system can be classified into four cases.

case 1:- undamped system, $\zeta = 0$

case 2:- under damped system $0 < \zeta < 1$

case 3:- critically damped system $\zeta = 1$

case 4:- over damped system $\zeta > 1$

The characteristic equation of the 2nd order system.

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$$

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2-1)}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

Where $\xi = 0$, $s_1, s_2 = \pm j\omega_n$; $\rightarrow$ roots are purely imaginary.

System is undamped.

$\xi = 1$, $s_1, s_2 = -\omega_n$ $\rightarrow$ roots are real & equal.

System is critically damped.

$\xi > 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ $\rightarrow$ roots are real & unequal.

System is overdamped.

$0 < \xi < 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}.$$

$$= -\xi\omega_n \pm j\omega_d$$ $\rightarrow$ roots are complex conjugate.

System is underdamped.

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$\omega_d \rightarrow$ damped frequency of oscillation of the system.

case 1: undamped system $(\zeta = 0)$ for unit step input (1)

The closed loop T.F. of 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Sub $\zeta = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}.$$

When input is unit step. $R(s) = 1/s$.

$$C(s) = \frac{1}{s} \frac{\omega_n^2}{(s^2 + \omega_n^2)}.$$

$$C(s) = \frac{A}{s} + \frac{Bs}{(s^2 + \omega_n^2)}. \rightarrow (1)$$

$$\frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A(s^2 + \omega_n^2) + Bs^2}{s(s^2 + \omega_n^2)}$$

$$\omega_n^2 = A(s^2 + \omega_n^2) + Bs^2$$

put $s = 0$

$$\omega_n^2 = A\omega_n^2$$

$$\boxed{A = 1}$$

$s^2 = -\omega_n^2$

$$\omega_n^2 = B(-\omega_n^2)^2$$

$$\boxed{\cancel{B = -1}}$$

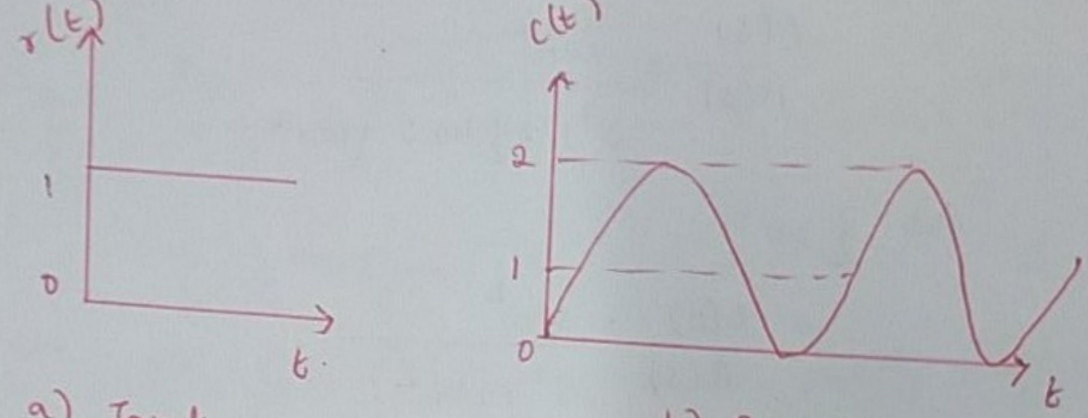
$$B = -1$$

$$C(s) = \frac{1}{s} - \frac{s}{(s^2 + \omega_n^2)}$$

$$L^{-1}\left\{\frac{s}{s^2 + \omega_n^2}\right\} = \cos\omega_n t$$

Time domain response

$$c(t) = L^{-1}(C(s)) = 1 - \cos\omega_n t.$$



a). Input b) Response.

the response is completely oscillatory.

$\therefore$ unit step response $= 1 - \cos\omega_n t$

step response $= A(1 - \cos\omega_n t)$.

case 2: under damped system $0 < \zeta < 1$ for unit step input

Transfer Function of 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

from characteristic equation.

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_d.$$ $\rightarrow$ roots are complex conjugate.

$$C(s) = \frac{R(s)\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$R(s) = 1/s$ (step input).

$$C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \rightarrow (1)$$

$$\omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + (Bs + C)s$$

$$\omega_n^2 = As^2 + A2\zeta\omega_n s + A\omega_n^2 + Bs^2 + Cs$$

equating s

$$2\zeta\omega_n A + C = 0.$$

equating s^2

$$A + B = 0.$$

const

$$\omega_n^2 = A\omega_n^2$$

$$\therefore \boxed{A = 1}$$

$$\boxed{C = -2\zeta\omega_n}$$

$$1 + B = 0$$

$$\boxed{B = -1}$$

$$C(s) = \frac{1}{s} + \frac{-s - 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s^2 + (\zeta\omega_n + j\omega_d))(s + (\zeta\omega_n}$$

$$(s + \zeta\omega_n)^2 + \omega_d^2.$$

$$= \frac{1}{s} - \frac{(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Multiply 3rd by ω_d

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n\,\omega_d}{\omega_d\left[(s+\zeta\omega_n)^2+\omega_d^2\right]}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]}$$

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

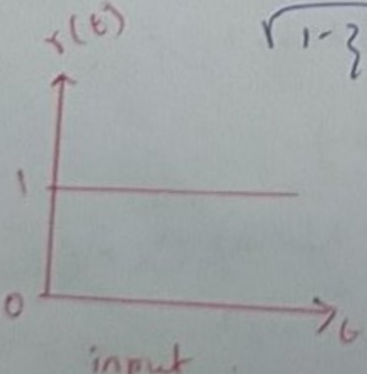
$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$

1, θ, $\sqrt{1-\zeta^2}$, ζ

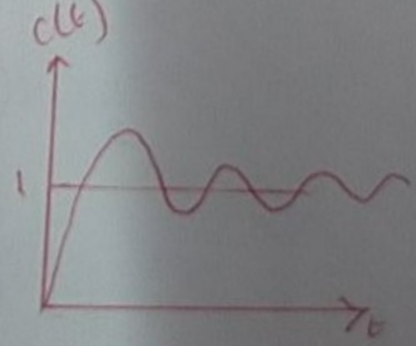
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

unit step response.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_d t+\theta)\right]$$



input



Response

Step response $c(t) = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta);\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$.

case 3: critically damped system $\zeta = 1$

Transfer function $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\zeta = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

for step i/p
$R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2}$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2} \rightarrow (1)$$

$$\omega_n^2 = A(s+\omega_n)^2 + B\,s(s+\omega_n) + Cs$$

equate s^2, s, const

$$\omega_n^2 = A(s^2 + \omega_n^2 + 2\omega_n s) + B(s^2 + s\omega_n) + Cs$$

s^2: $0 = A + B$

$\boxed{B = -1}$

const: $\omega_n^2 = A\omega_n^2$

$\boxed{A = 1}$

s: $0 = 2\omega_n A + B\omega_n + C$

$2\omega_n - \omega_n + C = 0$

$\boxed{C = -\omega_n}$

$$C(s) = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}$$

The time response in time domain

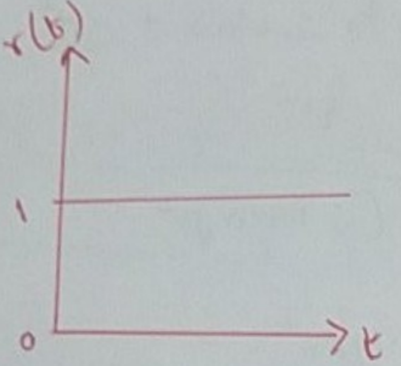
$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

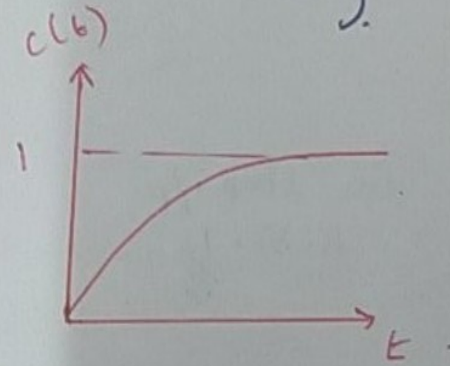
$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

unit step response

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t).$$

step response $= A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right].$





Response have No oscillation.

case 4: over damped System. $\zeta > 1$ for step input.

Transfer function $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$

$\zeta > 1$, roots are.

$$s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}.$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}.$$

$R(s) = 1/s$

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

(11)

$$\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{(s+s_1)} + \frac{C}{(s+s_2)} \rightarrow (1)$$

$$\omega_n^2 = A(s+s_1)(s+s_2) + Bs(s+s_2) + Cs(s+s_1)$$

$$\omega_n^2 = A(s^2 + s_2 s + s_1 s + s_1 s_2) + B(s^2 + ss_2) + C(s^2 + ss_1)$$

equating const, s, s^2.

$\omega_n^2 = A s_1 s_2$

$A = \dfrac{\omega_n^2}{s_1 s_2}$.

s^2

$A + B + C = 0$

$\dfrac{\omega_n^2}{s_1 s_2} + B + C = 0$.

s

$A(s_2 + s_1) + Bs_2 + Cs_1$

(or)

$s = -s_1$

$$\omega_n^2 = B(-s_1)(-s_1+s_2)$$

$$\boxed{\frac{\omega_n^2}{-s_1(s_2-s_1)} = B}$$

$s = -s_2$

$$C = \frac{\omega_n^2}{(-s_2)(-s_2+s_1)} = \frac{\omega_n^2}{-s_2(s_1-s_2)}.$$

$$C(s) = \frac{\omega_n^2}{s_1 s_2 s} - \frac{\omega_n^2}{s_1(s_2-s_1)(s+s_1)} - \frac{\omega_n^2}{s_2(s_1-s_2)(s+s_2)}$$

$$s_1 s_2 = \left(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)$$

$$= (\zeta\omega_n)^2 - \left(\omega_n\sqrt{\zeta^2-1}\right)^2$$

$$= \zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1) = \zeta^2\omega_n^2 - \omega_n^2\zeta^2 + \omega_n^2 = \omega_n^2$$

$$s_2 - s_1 = \cancel{\zeta\omega_n} + \omega_n\sqrt{\zeta^2-1} - \cancel{\zeta\omega_n} + \omega_n\sqrt{\zeta^2-1}$$

$$\boxed{s_2 - s_1 = 2\omega_n\sqrt{\zeta^2-1}}$$

$$s_1 - s_2 = \cancel{\zeta\omega_n} - \omega_n\sqrt{\zeta^2-1} - \cancel{\zeta\omega_n} - \omega_n\sqrt{\zeta^2-1}$$

$$\boxed{s_1 - s_2 = -2\omega_n\sqrt{\zeta^2-1}}$$

$$c(s) = \frac{\omega_n^2}{\omega_n^2 s} - \frac{\omega_n^2}{s_1(s+s_1)\,2\cancel{\omega_n}\sqrt{\zeta^2-1}} + \frac{\omega_n^2}{s_2(s+s_2)\,2\omega_n\sqrt{\zeta^2-1}}$$

$$= \frac{1}{s} - \frac{\omega_n}{s_1(s+s_1)2\sqrt{\zeta^2-1}} + \frac{\cancel{\omega_n}}{s_2\,2\cancel{\omega_n}\sqrt{\zeta^2-1}(s+s_2)}$$

Taking inverse Laplace Transform

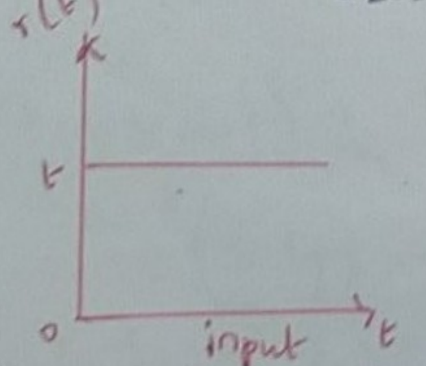
$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]$$

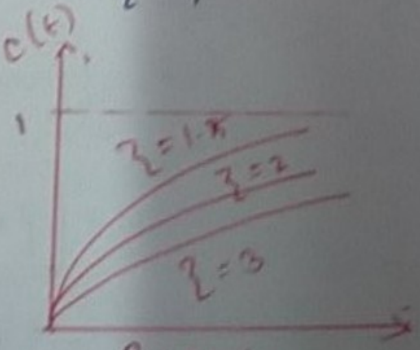
Step response $= A\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]\right]$

Where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$



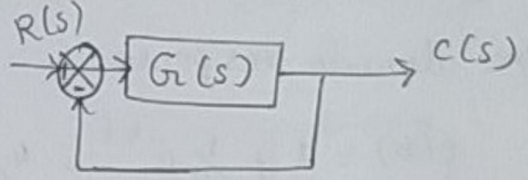
input



Response

→. No oscillation but it reach final steady value after long time.

Example 1: obtain the response of unity feedback system whose open Loop Transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.



Solution:

closed Loop T.F $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$= \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}} = \frac{4}{s^2+5s+4}$$

$$\frac{C(s)}{R(s)} = \frac{4}{(s+4)(s+1)}.$$

unit step Response $R(s) = 1/s$

$$C(s) = \frac{4}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+1)}$$

$$4 = A(s+4)(s+1) + B\,s(s+1) + C(s+4)s$$

put

$s=0$, $4 = A(4)$, $A = 1$

$s=-1$, $4 = C(3)(-1)$

$C = -4/3$

$s=-4$, $4 = B(-4)(-3)$

$B = 1/3$

$$C(S) = \frac{1}{S} + \frac{\frac{1}{3}}{(S+4)} - \frac{4/3}{(S+1)}$$

Taking inverse L.T.

:

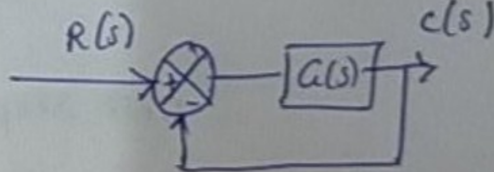
the time domain response

$$c(t) = 1 + \tfrac{1}{3} e^{-4t} - \tfrac{4}{3} e^{-t}$$

$$c(t) = 1 + \tfrac{1}{3}\left[e^{-4t} - 4e^{-t}\right]$$

Ex. 2 The unity feedback system is characterized by an open loop transfer function $G(S) = \frac{k}{S(S+10)}$. determine the gain k, so the system will have a damping ratio of 0.5 for this value of k. determine settling time, peak overshoot & time at peak overshoot for a unit step input.

Soln:



unity feedback.

Closed loop T.F $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)}$

$$= \frac{\frac{k}{S(S+10)}}{1 + \frac{k}{S(S+10)}} = \frac{k}{S^2+10S+k}.$$

Standard form of second order Transfer function

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{k}{S^2+10S+k}$$

Comparing.

$$\omega_n^2 = k$$
$$\omega_n = \sqrt{k}$$

$$2\zeta\omega_n = 10$$
$$\zeta = 0.5$$
$$2 \times 0.5 \times \sqrt{k} = 10$$
$$\sqrt{k} = 10$$
$$\boxed{k = 100}$$

$$\boxed{\omega_n = 10}$$

percentage peak overshoot $\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$

$$= e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} \times 100$$

$$= 16.3\%$$

peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

Ans:

$k = 100$

$\% M_p = 16.3\%$

$t_p = 0.363$ sec.

Type number of control system.

Loop Transfer function in terms of s.

$$G(s)H(s) = k\frac{P(s)}{Q(s)} = k\frac{(s+z_1)(s+z_2)\cdots}{s^N(s+P_1)(s+P_2)\cdots}$$

Where,

$z_1, z_2, z_3 \rightarrow$ zeros of Transfer function

$P_1, P_2, P_3 \rightarrow$ poles of Transfer function

$k \rightarrow$ constant

$N \rightarrow$ type number of System.

If N=0, then system is type-0 system.

N=1, then system is type-1 system.

N=2, then system is type-2 system.

N=3, then system is type-3 system, and so on

Steady state error:-

The steady state error is the value of error signal e(t) when t tends to infinity.

→ It is a measure of system accuracy.

Errors arises from

→ Nature of input

→ type of system

→ Non Linearity of system component.

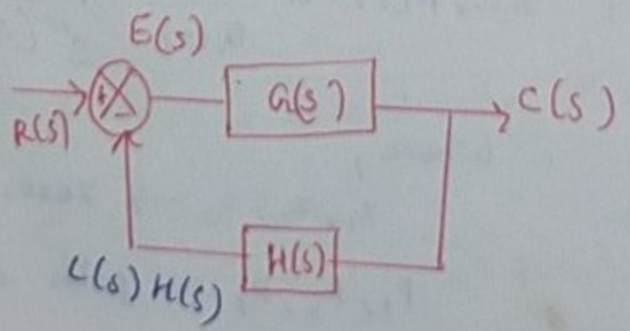
Steady state performance of stable control system is

→ Steady state error to step input

→ Steady state error to ramp input

→ Steady state error to parabolic input.

Consider a closed Loop system.



Let,

$R(s) \rightarrow$ input signal

$E(s) \rightarrow$ Error signal

$C(s)H(s) \rightarrow$ Feed back signal

$C(s) \rightarrow$ output signal (or) response.

Error signal $E(s) = R(s) - C(s)H(s) \rightarrow (1)$

output signal $C(s) = E(s) \cdot G(s) \rightarrow (2)$

Sub (2) in (1)

$$E(s) = R(s) - E(s)G(s)H(s)$$

$$E(s)[1 + G(s)H(s)] = R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Taking inverse L.T

$$e(t) = L^{-1}\left[\frac{R(s)}{1 + G(s)H(s)}\right]$$

$e_{ss} \rightarrow$ steady state error.

The steady state error is defined as the value of $e(t)$ when t tends to infinity.

$$e_{ss} = Lt_{t \to \infty} e(t)$$

The final value theorem of Laplace Transform

if $F(s) = L\{f(t)\}$ then $Lt_{t \to \infty} f(t) = Lt_{s \to 0} sF(s)$

using final value theorem,

The steady state error $e_{ss} = \underset{t \to \infty}{Lt}\ e(t) = \underset{s \to 0}{Lt}\ sE(s)$

$$= \underset{s \to 0}{Lt}\ \frac{sR(s)}{1+G(s)H(s)}$$

Static error constant :-

When the control system is excited with standard input signal, the steady state error may be zero, constant or infinity.

→ The value of steady state error depends on the type number & input signal.

Input is step signal → Type 0 system → constant steady state error.

positional error constant $\boxed{k_p = \underset{s \to 0}{Lt}\ G(s)H(s)}$

Input is Ramp signal (or) velocity → Type 1 system → "

$$\boxed{k_v = \underset{s \to 0}{Lt}\ sG(s)H(s)}$$

(velocity error constant)

parabolic signal (or) acceleration → Type 2 system → const. steady state.

$$\boxed{k_a = \underset{s \to 0}{Lt}\ s^2 G(s)H(s)}$$

(acceleration error constant).

k_p, k_v, k_a are in general called static error constants.

1) Steady state error when the input is unit step signal.

Steady state error $e_{ss} = \underset{s \to 0}{Lt} \dfrac{s\,R(s)}{1 + G(s)H(s)}$.

When the input is unit step

$$R(s) = 1/s.$$

$$e_{ss} = \underset{s \to 0}{Lt} \frac{s \times \frac{1}{s}}{1 + G(s)H(s)} = \underset{s \to 0}{Lt} \frac{1}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + \underset{s \to 0}{Lt}\, G(s)H(s)} = \frac{1}{1 + k_p}.$$

Where, $k_p \Rightarrow \underset{s \to 0}{Lt}\, G(s)H(s)$.

$k_p \to$ positional error constant.

Type-0 system :-

$$k_p = \underset{s \to 0}{Lt}\, G(s)H(s) = \underset{s \to 0}{Lt}\, \frac{k(s+z_1)(s+z_2)\ldots}{(s+P_1)(s+P_2)\ldots}$$

$$= \frac{k\, z_1 z_2 z_3 \ldots}{P_1 P_2 P_3}$$

$$= \text{constant}.$$

$$e_{ss} = \frac{1}{1+k_p} = \text{constant}.$$

Type-1 system :-

$$k_p = \underset{s \to 0}{Lt}\, G(s)H(s) = \underset{s \to 0}{Lt}\, \frac{k(s+z_1)(s+z_2)\ldots}{s(s+P_1)(s+P_2)\ldots}$$

$$= \infty$$

$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+\infty} = 0$$

2) Steady state error when input is unit ramp signal

Steady state error $e_{ss} = Lt_{s\to 0} \frac{SR(s)}{1+G(s)H(s)}$.

unit Ramp $R(s) = \frac{1}{s^2}$

$$e_{ss} = Lt_{s\to 0} \frac{s \times \frac{1}{s^2}}{1+G(s)H(s)} = Lt_{s\to 0} \frac{1}{s + sG(s)H(s)}$$

$$= \frac{1}{Lt_{s\to 0} sG(s)H(s)} = \frac{1}{k_v}$$

Where,

$k_v \rightarrow Lt_{s\to 0} sG(s)H(s)$

$k_v \rightarrow$ velocity error constant.

Type-0 System

$$k_v = Lt_{s\to 0} sG(s)H(s) = Lt_{s\to 0} s\frac{k(s+z_1)(s+z_2)\ldots}{(s+p_1)(s+p_2)\ldots}$$

$$= 0$$

$$e_{ss} = \frac{1}{k_v} = \frac{1}{0} = \infty$$

Type-1 System.

$$k_v = Lt_{s\to 0} sG(s)H(s) = Lt_{s\to 0} \frac{sk(s+z_1)(s+z_2)\ldots}{s(s+p_1)(s+p_2)}$$

$$= \frac{k\, z_1, z_2, z_3 \ldots}{p_1, p_2, p_3} = \text{constant}$$

$$e_{ss} = \frac{1}{k_v} = \text{constant}.$$

Type - 2 system:

$$k_v = Lt_{s \to 0} s\,G(s)H(s) = Lt_{s \to 0} \cancel{s}k \frac{(s+z_1)(s+z_2)(s+z_3)}{s^{\cancel{2}}(s+P_1)(s+P_2)\cdots}$$

$$= \infty$$

$$e_{ss} = \frac{1}{k_v} = \frac{1}{\infty} = 0$$

3) Steady state error when the input is unit parabolic signal.

Steady state error $e_{ss} = Lt_{s \to 0} \frac{s\,R(s)}{1 + G(s)\,H(s)}$.

When the input is unit parabola $R(s) = \frac{1}{s^3}$

$$e_{ss} = Lt_{s \to 0} \frac{\cancel{s}\,\frac{1}{s\cancel{s}^2}}{1 + G(s)H(s)}$$

$$= Lt_{s \to 0} \frac{1}{s^2(1 + G(s)\,H(s))}$$

$$= \frac{1}{Lt_{s \to 0}(s^2 G(s)\,H(s))}.$$

$$= \frac{1}{k_a}.$$

Where, $k_a \to Lt_{s \to 0} s^2 G(s)\,H(s)$

$\to$ acceleration error constant

Type 0 system:-

$$k_a \to Lt_{s \to 0} s^2 G(s)\,H(s) = Lt_{s \to 0} s^2 \frac{k\,(s+z_1)(s+z_2)}{(s+P_1)(s+P_2)\cdots}$$

$$= 0$$

$$e_{ss} = 1/K_a = 1/0 = \infty$$

Type 1 - system

$$K_a = \lim_{s\to 0} s^2 G(s) H(s) = \lim_{s\to 0} \frac{s^{\cancel{2}} k(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+p_1)(s+p_2)\ldots}$$

$$= 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Type 2 system

$$K_a = \lim_{s\to 0} s^2 G(s) H(s) = \lim_{s\to 0} \frac{\cancel{s^2} k(s+z_1)(s+z_2)\ldots}{\cancel{s^2}(s+p_1)(s+p_2)}$$

$$= \text{constant}$$

$$e_{ss} = 1/K_a = \text{constant}.$$

Type 3 system:

$$K_a = \lim_{s\to 0} s^2 G(s) H(s)$$

$$= \lim_{s\to 0} \frac{\cancel{s^2} k(s+z_1)(s+z_2)\ldots}{s^{\cancel{3}}(s+p_1)(s+p_2)\ldots}$$

$$= \infty$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

Static error constant for various type number of system

Error Constant	Type Number of System			
	0	1	2	3
k_p	constant	∞	∞	∞
k_v	0	Constant	∞	∞
k_a	0	0	Constant	∞

Steady state error for various type of inputs.

Input signal	Type Number of system			
	0	1	2	3
unit step	$\frac{1}{1+k_p}$	0	0	0
unit Ramp	∞	$\frac{1}{k_v}$	0	0
unit parabolic	∞	∞	$\frac{1}{k_a}$	0

Drawback;

→ Static error coefficient doesn't show the variation of error with time.

→ input should be standard input.

Generalized error coefficient.

The generalised error coefficient gives the steady state error as a function of time.

→ Steady state error can be found for any type of input.

The error signal $E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{1}{1+G(s)H(s)} R(s)$

$$E(s) = F(s) \cdot R(s) \quad \rightarrow (1)$$

Where,

$$F(s) = \frac{1}{1+G(s)H(s)}$$

The convolution theorem of Laplace Transform states that the Laplace Transform of the convolution of two time domain signals is equal to the product of their individual Laplace transform.

eq (1) inverse Laplace transform

$$e(t) = f(t) * r(t) \quad \rightarrow (2)$$

By convolution

$$f(t) * r(t) = \int_{-\infty}^{\infty} f(T)\, r(t-T)\, dT \quad \rightarrow (3)$$

Where, T → dummy variable

sub (3) in (2)

$$e(t) = \int_{-\infty}^{\infty} f(T)\, r(t-T)\, dT$$

Assume, the input signal starts only at $t=0$.

$$e(t) = \int_0^t f(T)\, r(t-T)\, dT \rightarrow (4)$$

By taylor's series,

the signal $r(t-T) = r(t) - T\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) \cdots + (-1)^n \frac{T^n}{n!} r(t)$ $\rightarrow (5)$

Where,
$\dot{r}(t) \rightarrow$ 1st derivative of $r(t)$
$\ddot{r}(t) \rightarrow$ 2nd ,,

Sub (5) in (4)

$$e(t) = \int_0^t f(T)\left[r(t) - T\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + \cdots (-1)^n \frac{T^n}{n!} r(t) \cdots\right] dT$$

$$= \int_0^t f(T)\, r(t)\, dT - \int_0^t f(T)\, T\dot{r}(t)\cdot dT + \int_0^t f(T) \frac{T^2}{2!}\ddot{r}(t)\, dT \cdots\cdots \infty$$

$$e(t) = r(t)\int_0^t F(T)\,dt - \dot{r}(t)\int_0^t T F(T)\, dT + \ddot{r}(t)\int_0^t \frac{T^2}{2} f(T)\, dT \cdots$$

Let, $c_0 = \int_0^t F(T)\,dT$, $c_1 = -\int_0^t T F(T)\, dT$.

$$c_2 = \int_0^t T^2 f(T)\,dT \;\ldots\ldots\; c_n = (-1)^n \int_0^t T^n f(T)\,dT$$

$$e(t) = r(t)c_0 + \dot{r}(t)c_1 + \ddot{r}(t)\frac{c_2}{2!} + \dddot{r}(t)\frac{c_3}{3!}\ldots$$

$$= c_0 r(t) + c_1\dot{r}(t) + \frac{c_2}{2!}\ddot{r}(t) + \frac{c_3}{3!}\dddot{r}(t)\ldots$$

Where,

$c_0, c_1, c_2 \ldots c_n \rightarrow$ generalised error coefficient.

(or)

dynamic error coefficient.

The steady error is obtained by taking Limit $t \to \infty$ on $e(t)$

$$e_{ss} = \lim_{t\to\infty} r(t)c_0 + \dot{r}(t)c_1 + \ddot{r}(t)\frac{c_2}{2!}\ldots$$

Evaluation of Generalised error coefficient.

The generalized error coefficient.

$$c_n = (-1)^n \int_0^t T^n f(T)\,dT$$

Where,

$$F(s) = \frac{1}{1+G(s)H(s)}.$$

$$c_0 = \lim_{s\to 0} F(s)$$

$$c_1 = \lim_{s\to 0} \frac{d}{ds}F(s).$$

$$c_2 = \lim_{s\to 0} \frac{d^2}{ds^2}F(s).$$

Hlly

Similarly

$$c_n = Lt_{s \to 0} \frac{d^n}{ds^n} F(s).$$

correlation between static & dynamic error coefficient

$$c_0 = \frac{1}{1+k_p}$$

$$c_1 = \frac{1}{k_v}$$

$$c_2 = \frac{1}{k_a}.$$

problem 1: Find the unity feedback control system the open Loop Transfer Function $G(s) = \frac{10(s+2)}{s^2(s+1)}$. find the

(a) position, velocity, Acceleration error constant.

(b) the steady state error when the input is R(s)

Where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Soln:

a) static error constant.

unity feed back system $H(s) = 1$

position error constant $k_p = Lt_{s \to 0} G(s) H(s)$

$$= Lt_{s \to 0} \frac{10(s+2)}{s^2(s+1)}$$

$$= \infty$$

velocity error constant $k_v = Lt_{s \to 0} s G(s) H(s)$

$$= Lt_{s \to 0} s \frac{10(s+2)}{s^2(s+1)}$$

$$= \infty$$

acceleration error constant $k_a = \lim_{s \to 0} s^2 G(s) H(s)$

$$= \lim_{s \to 0} s^2 G(s)$$

$$= \lim_{s \to 0} s^2 \times \frac{10(s+2)}{s^2(s+1)}$$

$$= \frac{10 \times 2}{1} = 20.$$

b) To find steady state error.

Steady state error for non standard input.

error signal, $e(t) = r(t) C_0 + C_1 \dot{r}(t) + \frac{C_2}{2!} \ddot{r}(t) \ldots$

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

Inverse L.T

$$r(t) = \mathcal{L}^{-1}[R(s)\} = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right]$$

$$= 3 - 2t + \frac{1}{3}\frac{t^2}{2!}$$

$$r(t) = 3 - 2t + \frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt} r(t) = -2 + \frac{1}{6} 2t = -2 + \frac{1}{3} t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2} r(t) = \frac{d}{dt} \dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3} r(t) = \frac{d}{dt} \ddot{r}(t) = 0$$

The generalized error constant

$$C_0 = \lim_{s \to 0} F(s)$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds} F(s)$$

$$C_2 = \lim_{s \to 0} \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)}.$$

$$= \frac{1}{1 + \frac{10(s+2)}{s^2(s+7)}}$$

$$F(s) = \frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0 = \lim_{s\to 0} F(s) = \lim_{s\to 0} \frac{s^3+s^2}{s^3+s^2+10s+20} = 0$$

$$C_1 = \lim_{s\to 0} \frac{d}{ds} F(s) = \lim_{s\to 0}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0.$$

$$C_2 = \lim_{s\to 0} \frac{d^2}{ds^2} F(s) = \lim_{s\to 0}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40) - (20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

error signal, $e(t) = C_0\, r(t) + C_1\, \dot r(t) + \frac{C_2}{2!}\, \ddot r(t)$

$$= \left(1 - 2t + \frac{t^2}{6}\right)\times 0 + \left(-2 + \frac{t}{3}\right)\times 0 + \frac{1}{3}\times\frac{1}{10}\times\frac{1}{2!}$$

$$= \frac{1}{60}.$$

Steady state error, $e_{ss} = \lim_{t\to\infty} e(t)$

$$= \lim_{t\to\infty} \frac{1}{60}$$

$$= \frac{1}{60}$$

Effects of P, PI, PID controllers modes of feedback control

In feedback control systems, controller may be introduced to modify the error signal & to achive better control action.

→ Modify the transient response & steady state error of the system.

Effect of proportional controller (P- controller).

The proportional controller is a device that produces a output signal $u(t)$ which is proportional to input error signal $e(t)$.

P controller

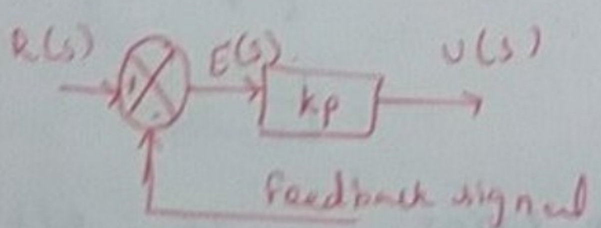
$$u(t) \propto e(t).$$
$$= k_p \, e(t)$$

Where k_p → proportional gain or constant.

Taking L.T.

$$U(s) = k_p E(s)$$

$$\frac{U(s)}{E(s)} = k_p$$



The proportional controller amplifies the error signal & increase the loop gain of the

PI controller: proportional plus Integral controller.

PI controller produces an o/p s/g.

consist of 2 terms.

1) proportional to error signal.

2) proportional to integral of error s/g.

$$u(t) \propto [e(t) + \int e(t)\,dt)$$

$$= k_p\, e(t) + \frac{k_p}{T_i} \int e(t)\,dt$$

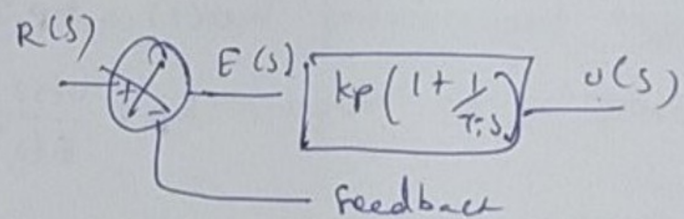
T_i → integral time.

Laplace L.T.

$$u(s) = k_p E(s) + \frac{k_p}{T_i s} E(s)$$

$$= E(s)\left[k_p + \frac{k_p}{T_i s}\right]$$

$$\frac{U(s)}{E(s)} = k_p\left[1 + \frac{1}{T_i s}\right]$$



Adv: Loop gain ↑.

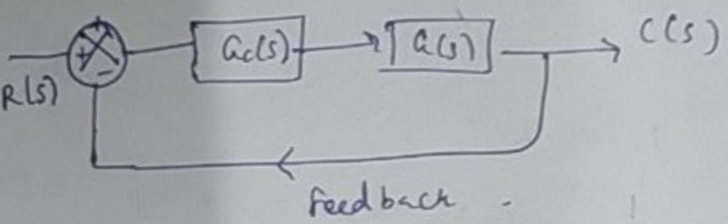
reduce steady state error.

effect: The transfer function of PI controller

controller → device which is introduce on the Feedback.

→ control the steady state & transient requirement.

P- controller: (proportional controller).



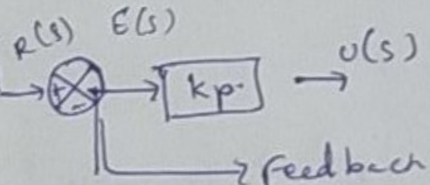
proportional controller is a device that produce a control signal $u(t)$ which is proportional to error signal

$$u(t) \propto e(t).$$
$$= k_p\, e(t)$$

T. L.

$$U(s) = k_p E(s).$$

$$k_p = \frac{U(s)}{E(s)}$$



effect:

→ p controller produces o/p signal $\propto$ error signal.

P-controller ↑ Loop gain

1) Steady state tracking accuracy

2). disturbance s/l rejection.

3) Relative stability.

drawback: → produce const. steady state error.

System. The following aspects of the system behaviour are improved by increasing Loop gain

(i) Steady state tracking accuracy,

(ii) Disturbance signal rejection.

(iii) Relative stability.

disadvantage:

P-controller produces a constant steady state error.

Effect of PI controller.

the proportional plus integral controller (PI controller) produces an output signal consisting of two terms.

1. proportional to error signal.
2. proportional to integral of error signal.

$$u(t) \propto \left[e(t) + \int e(t)\,dt \right]$$

$$u(t) = k_p\, e(t) + \frac{k_p}{T_i} \int e(t)\,dt$$

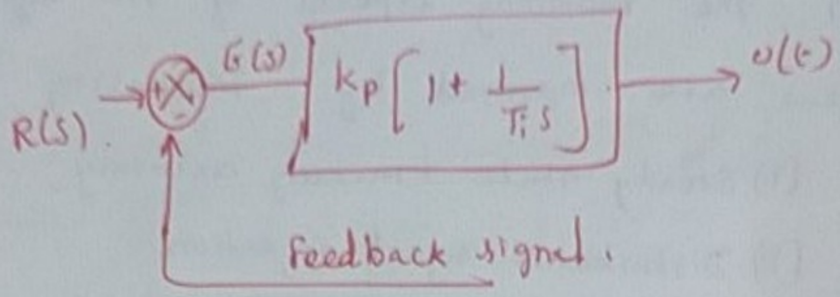
Where, k_p - proportional gain

T_i - Integral time.

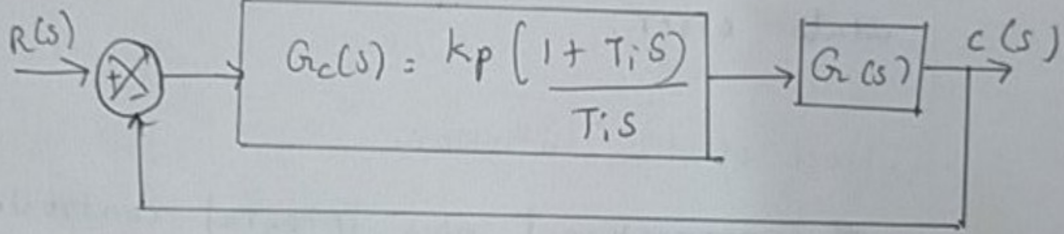
Taking L.T

$$U(s) = k_p E(s) + \frac{k_p}{T_i} \frac{E(s)}{s}$$

$$\frac{U(s)}{E(s)} = k_p \left[1 + \frac{1}{T_i s} \right]$$



The block diagram of unity feedback system with PI controller.



$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}.$$

Loop T/f $= G_c(S)\,G(s)\,H(s) = G_c(S)\,G(s)$

$$= k_p\left(\frac{1+T_iS}{T_iS}\right) \times \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

Closed Lood. T/f. $\frac{C(S)}{R(S)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}$

$$= \frac{k_i\omega_n^2(1+T_iS)}{s^2+2\zeta\omega_n s^2+k_p\omega_n^2 s+k_i\omega_n^2}$$

$\frac{k_p}{T_i} = k_i$

Advantage:

proportional action → increase Loop gain & Less sensitive to variation of system parameter.

Integral action → eliminates the steady state error

disadvantage:

→. k_p affects both proportional & integral parts of control action.

The inverse of the integral time T_i is called the reset rate.

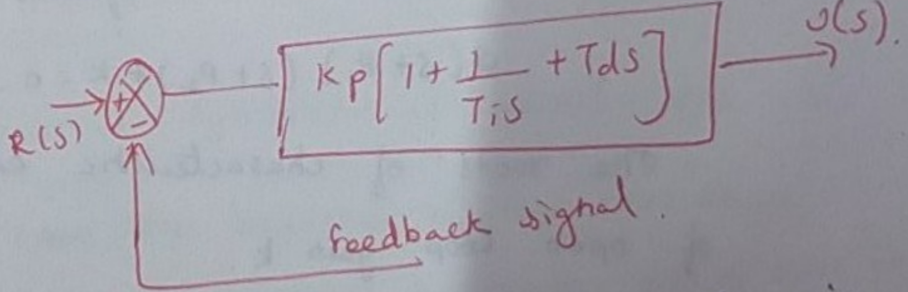
Effect of PID controller:-

It produces an output signal consisting three terms.

1. proportional to error signal
2. proportional to integral of error signal
3. proportional to derivative of error signal.

$$u(t) \propto e(t) + \int e(t)\,dt + \frac{d}{dt}e(t).$$

$$\frac{U(s)}{E(s)} = k_p\left[1 + \frac{1}{T_i s} + T_d s\right].$$



P (proportional) controller →. stabilizes the gain but produce a steady state error

I (Integral) controller → reduce steady state error

D (derivative) controller →. reduces the rate of change of error.

Root Locus:-

The Root Locus technique was introduced by W.R. Evans in 1948 for the analysis of control System.

→ It is a powerful tool for adjusting the location of closed Loop poles by varying the system parameters.

Open Loop Transfer function $G(s) = \frac{k}{s(s+P_1)(s+P_2)}$.

Closed Loop T/f. with unity feedback $\frac{C(s)}{R(s)} = \frac{\frac{k}{s(s+P_1)(s+P_2)}}{1+\frac{k}{s(s+P_1)(s+P_2)}}$

$$= \frac{k}{s(s+P_1)(s+P_2)+k}.$$

The denominator polynomial of $\frac{C(s)}{R(s)}$ is the characteristic equation of system.

$$s(s+P_1)(s+P_2)+k=0.$$

The roots of characteristic equation is a function of open Loop gain k.

When $k=0$, the roots are given by open Loop poles

$k=\infty$, the roots will take the value of open Loop zeros.

the path taken by the roots of characteristic equation when open loop gain k is varied from 0-∞ are called root Loci.

contruction of Root Locus.

→ The exact root Locus is sketched by trail & error procedure.

→ poles & zeros of G(s) H(s) are Located on the S-plane on graph sheet.

At trail point select, $S = S_a$.

→ determine the angles of vectors drawn from poles & zero to the trail point.

→ from the angle criterion, determine the angle to be contributed by these vectors to make the trail point as a point on root Locus.

→ Trail & error procedure for sketching root Locus is tedious.

Rules for construction of Root Locus

Rule 1: The root Locus is symmetrical about the axis.

Rule 2: The No. of ~~open loop~~ branches of the root Locus terminating on ∞ is equal to (n-m) No. of open Loop poles - zeros.

Rule 3: Each branch of the root Locus originates from an open Loop pole at k=0 & terminates at open Loop zeros corresponding to $k = \infty$.

Rule 4: A point on the real axis lies on the Locus, if the No. of. open Loop poles & zeroes on the real axis the right of this point is odd.

Rule 5: The $(n-m)$ root Locus branches that tends to ∞. a straight line asymptotes making angles with the real axis.

$$\varphi_A = \frac{180^\circ (2q+1)}{n-m} ; \; q = 0, 1, 2 \dots n-m.$$

Rule 6: The point of intersection of the asymptoes with a real axis is at $S = \sigma_A$.

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}.$$

Rule 7: Breakaway & break in point of the root Locus are determined from the roots of the equation

$$\frac{dk}{ds} = 0.$$

If r number of branches of root Locus meet at a point, then they break away at an angle of $\pm \frac{180^\circ}{r}$.

Rule 8: The angle of departure from a complex open Loop pole is given by

$$\varphi_p = \pm 180^\circ (2q+1) + \varphi \; , \; q = 0, 1, 2 \dots$$

open Loop zero $\quad \varphi_z = \pm 180^\circ (2q+1) + \varphi$

$$q = 0, 1, 2.$$

Rule 8: The point of intersection of root Locus branch with imaginary axis can be determined by Routh criterion.

(or)

put $s = j\omega$ in the characteristic equation.

Equate real & imaginary part to zero, to Solve for ω & k.

$\omega \rightarrow$ intersection points on imaginary axis.

$k \rightarrow$ value of gain at the intersection point.

Rule 9: The open Loop gain k at any point $S = S_a$ on the root Locus is given by

$$k = \frac{\prod_{i=1}^{n} |S_a + P_i|}{\prod_{i=1}^{m} |S_a + Z_i|} = \frac{\text{product of vector Length from open Loop poles to the point } S_a}{\text{product of vector Length from open Loop zeros to the point } S_a}.$$

Procedure for constructing Root Locus.

Step 1: Location of poles & zeros.

draw the real & imaginary axis on an ordinary graph sheet. choose same Scale for both real & imaginary.

$\times \rightarrow$ pole.

$o \rightarrow$ zero

The origin of a root Locus is at a pole & the end is at a zero.

$n \rightarrow$ No. of poles.

$m \rightarrow$ No. of finite zeros.

Step 2: $(n-m)$ root branches will end at zero at ∞.

Root Locus on real axis.

If the total (No. of poles) & zeros on the real axis of the test point is

odd number $\rightarrow$ test point Lies on the root Locus.

even number $\rightarrow$ doesn't Lie on Root Locus

Step 3: Angle of asymptotes & centroid

No. of asymptotes = No. of root Locus branches going to ∞

$$\text{Angle of asymptotes} = \pm \frac{180(2q+1)}{n-m}$$

$$q = 0, 1, 2 \ldots (n-m)$$

Centroid (meeting point of asymptote with real axis) $= \dfrac{\text{Sum of pole} - \text{Sum of Zeros}}{n-m}$.

Step 4: Breakaway & Breakin points.

It Lie either in real axis (or) exist as complex conjugate pairs.

If there is a root Locus on real axis between two poles $\rightarrow$ breakaway point.

Root Locus on real axis between 2 zeros → Breakin point.

* If there is a root Locus on real axis between poles & zero then there exists may be or may not be breakaway (or) breakin point.

Let the characteristic equation be in the form

$$B(s) + k\,A(s) = 0$$

$$k = -\frac{B(s)}{A(s)}.$$

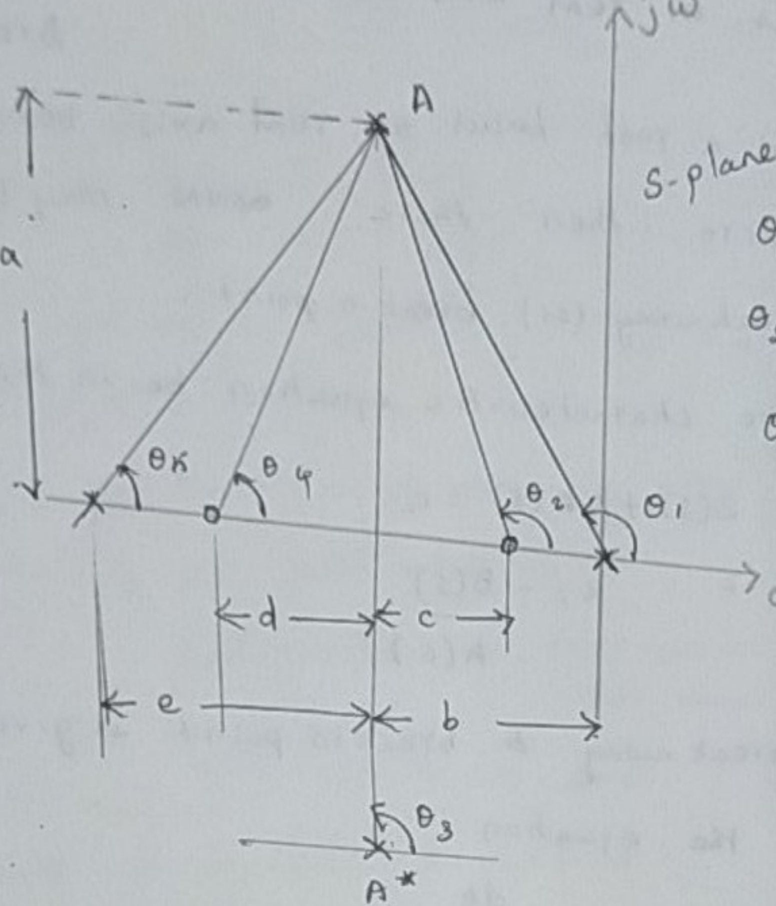
The breakaway & breakin point is given by roots of the equation

$$\frac{dk}{ds} = 0.$$

The gain k should be positive & real.

Step 5: Angle of departure & angle of arrival.

Angle of departure (from a complex pole A) $= 180^\circ - \left\{\begin{array}{l}\text{sum of angles}\\ \text{of vector to the}\\ \text{complex poles A}\\ \text{from other poles}\end{array}\right\} + \left\{\begin{array}{l}\text{sum of angles of vector to}\\ \text{the complex pole A from}\\ \text{zero}\end{array}\right\}$



$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$

$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$

$\theta_3 = 90°$

$\theta_4 = \tan^{-1} a/d$

$\theta_A = \tan^{-1} a/e.$

Angle of departure of pole $A = 180° - (\theta_1 + \theta_3 + \theta_A) + (\theta_2 + \theta_4)$

Angle of departure of pole $A^* = -$ [Angle of departure at pole A].

Step6: point of intersection of root Locus with imaginary axis.

There are 3 methods.

1) By Routh Hurwitz array
2) By Trail & error approach.
3) Let, $s = j\omega$ in the characteristic equation. separate real & imaginary part. & equate to zero.

Solve the equation find ω & K.

$\omega \rightarrow$ point where the root locus crosses imaginary axis.

$k \rightarrow$ gain at there crossing point.

Step 7: Test point & root locus.

→ choose a test point, using a protractor roughly estimate the angles of vectors drawn to this point. & adjust the point to satisfy the angle criterion.

→ Repeat the procedure for few more test points.

→ Sketch the root locus from the knowledge of sketches & information obtained in step 1 through 6.

Determination of open loop gain for a specified damping of the dominant roots.

dominant pole → complex conjugate pole.

high order system → dominant poles are very close to origin.

all other poles → Lying faraway from the dominant poles.

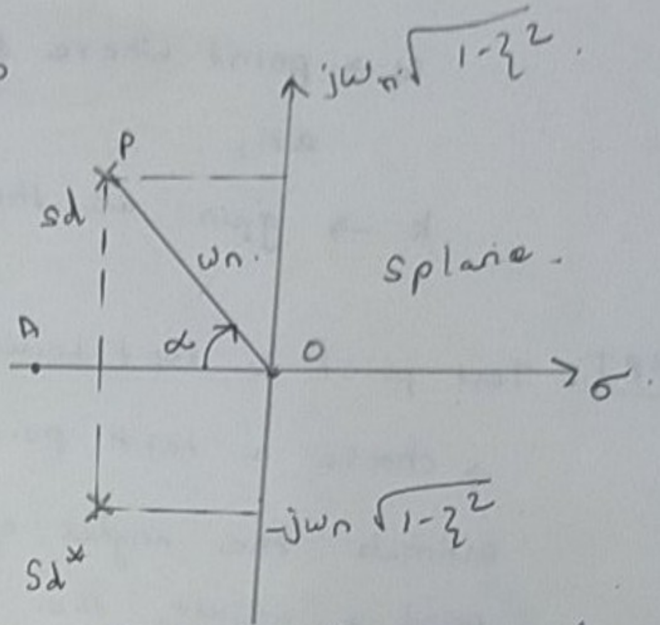
→ Less effect on transient response.

Transfer function of higher order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_d = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

In right angle triangle OAP



$$\cos\alpha = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\alpha = \cos^{-1}\zeta$$

$$k_{sd} : \frac{\text{Product of Length of vector from open Loop poles to dominant pole}}{\text{product of Length of vector from open Loop zeros to dominant pole.}}$$

Root Locus Techniques.

→ Stability Analysis.

→ Range of value of k for a stable system can be determined.

→ dominant roots are used to estimate the damping ratio & natural frequency of the system. ζ & ω_n time domain specification can be calculated.

problem: A unity feedback control system has an open loop Transfer function $G(s) = \frac{k}{s(s^2+4s+13)}$. sketch the root locus

(24)

Soln:

Step 1: To locate poles & zeros.

$$s(s^2+4s+13) = 0.$$

the poles are $s = 0$, $s^2+4s+13 = 0$.

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$$

$\therefore$ poles are $p_1 = 0$, $p_2 = -2+j3$, $p_3 = -2-j3$.

poles are marked by (X).

Step 2: To find the root locus on the real axis.

only one pole lie on the origin.

total No. of poles & zero ~~at~~ on the real axis is one. Which is odd number.

$\therefore$ Negative real axis will be part of root locus.

draw as bold line.

Step 3: To find the angle of asymptotes & centroid.

there are 3 poles. No. of root locus branches are three.

There is no finite zero, 3 root locus branches end at zero at ∞.

$$\text{Angle of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}, \quad q = 0, 1, 2 \ldots$$

$$q = 0, 1, 2, 3.$$

$n = 3, \ m = 0$

$$q = 0, \quad \text{angles} = \pm \frac{180^\circ}{3} = \pm 60^\circ$$

$$q = 1, \quad \text{angle} = \pm \left(\frac{180 \times 3}{3}\right) = \pm 180^\circ$$

$$q = 2, \quad \text{angle} = \pm \left(\frac{180^\circ \times 5}{3}\right) = \pm 300 = \mp 60^\circ$$

$$q = 3, \quad \text{angle} = \pm \left(\frac{180^\circ \times 7}{3}\right) = \pm 420^\circ = \pm 60^\circ$$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{sum of zero}}{n-m}$$

$$= \frac{0 - 2 + \cancel{j3} - 2 - \cancel{j3} - 0}{3} = -\frac{4}{3} = -1.33.$$

Step 4: To find break away & break in points.

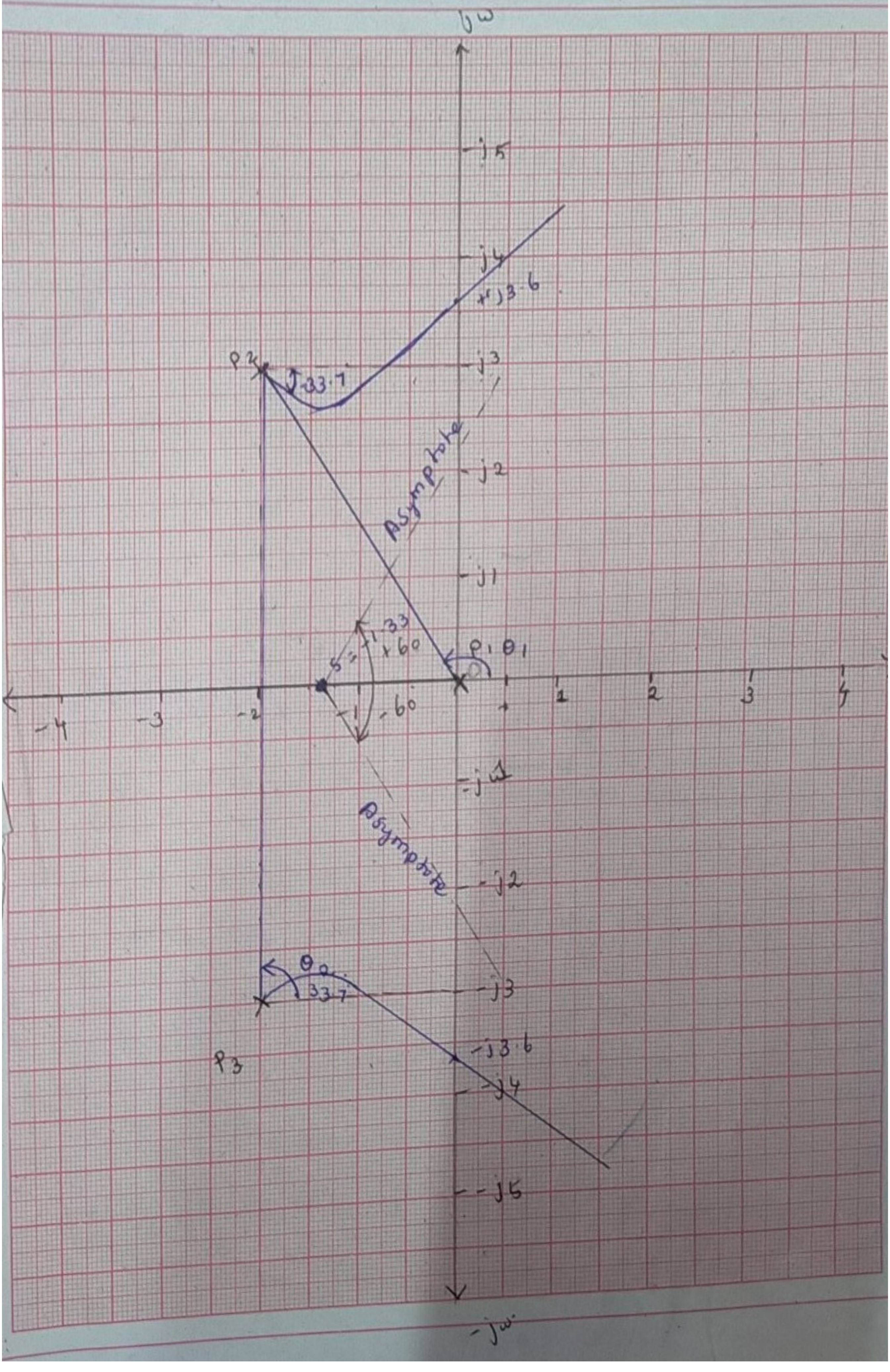
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{k}{s(s^2+4s+13)}}{1 + \frac{k}{s(s^2+4s+13)}}$$

$$= \frac{k}{s(s^2+4s+13) + k}.$$

the characteristic equation.

$$s(s^2+4s+13) + k = 0.$$

$$s^3 + 4s^2 + 13s + k = 0$$



jω
j5
j4
+j3.6
P2
j3
33.7
Asymptote
j2
j1
1.33
S = +60
P1 θ1
-4
-3
-2
-1
-60
1
2
3
4
-j1
Asymptote
-j2
θ2
33.7
-j3
-j3.6
P3
-j4
-j5
-jω

$k = -s^3 - 4s^2 - 13s$.

differentiate eqn.

$$\frac{dk}{ds} = -(3s^2 + 8s + 13)$$

$\frac{dk}{ds} = 0$. $\quad -(3s^2 + 8s + 13) = 0$

$$s = \frac{-8 \pm \sqrt{8^2 - 4 \times 13 \times 3}}{2 \times 3} = -1.33 \pm j1.6$$

When $s = -1.33 + j1.6$.

$$k = -(s^3 - 4s^2 - 13s) = -[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 - 13(-1.33 + j1.6)].$$

$\neq$ positive & real.

Since $s = -1.33 - j1.6 \rightarrow$ not real & positive.

$\therefore$ It is neither breakaway nor break in point.

Step 6: angle of departure.

draw the vector from all the other poles.

θ_1 & θ_2 of the vector.

$$\theta_1 = 180° - \tan^{-1}\left(\frac{3}{2}\right) = 123.7°$$

$$\theta_2 = 90°$$

angle of departure of complex pole $P_2 = 180° - (\theta_1 + \theta_2)$

$= 180° - (123.7° + 90°)$

$= -33.7°$.

$\therefore$ $P_3 = 33.7°$

Step 6: crossing point of imaginary axis.

$$s^3 + 4s^2 + 13s + k = 0.$$

put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + k = 0.$$

$$-j\omega^3 - 4\omega^2 + 13j\omega + k = 0$$

equating real & imaginary part.

$$-4\omega^2 + k = 0$$

$$k = 4\omega^2$$

$$= 4 \times 13$$

$$= 52$$

$$-\omega^3 + 13\omega = 0$$

$$\omega^2 = -13$$

$$\omega = \pm\sqrt{13} = \pm 3.6$$

$\therefore$ The crossing point of root Locus is $\pm j3.6$

$k = 52$. (This is the limiting value of k for the stability of the system).

6/4/2022 - 1, 5, ~~7~~ 10, 15, 17

7/4/2022 - 4, 5, ~~7~~, ~~10~~, 13,

Frequency Response

Frequency Response:-

The frequency domain Transfer function $T(j\omega)$ is a complex function of ω. The Magnitude and phase functions will be real functions of ω. They are called. frequency Response.

Frequency response
- → open Loop $G(s)H(s) \xrightarrow{s=j\omega} G(j\omega)H(j\omega)$ $= |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$
- → closed Loop

↓

$M(s) \xrightarrow{s=j\omega} M(j\omega) = |M(j\omega)| \angle M(j\omega)$

Advantage:

→ Absolute & relative stability of closed Loop system is estimated.

→ complicated system Transfer function is determined.

→ practical test for sinusoidal signal generator,

→ Effect of noise is correctly measured.

→ frequency response analysis & designs can be extended to certain nonLinear control systems

Frequency Response plots :-

Frequency Response analysis of control system can be carried either analytically or graphically.

Graphical techniques.

1. Bode plot
2. polar plot (Nyquist plot)
3. Nichols plot

} open Loop System.

4. M & N circles
5. Nichols chart

} closed Loop System.

→ It is used to determine the frequency domain specification.

→ Study the stability of the system.

→ Adjust the gain of the system to satisfy the desired specification.

<u>Bode plot</u>:

→ It is a frequency response plot of the sinusoidal Transfer function of the system.

→ It consists of two graphs

Magnitude Vs $\log \omega$

phase angle Vs $\log \omega$

<u>Adv</u>:

Multiplication of Magnitude can be converted into addition.

procedure for magnitude plot of Bode plot

Example: Sketch Bode plot for the following transfer function & determine the system gain k for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}$$

Soln:

Step 1: put $s = j\omega$

$$G(j\omega) = \frac{k(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Let $k = 1$

$$G(j\omega) = \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Step 2: Magnitude plot.

The corner frequency $\omega c_1 = \frac{1}{0.2} = 5$ rad/sec

$\omega c_2 = \frac{1}{0.02} = 50$ rad/sec.

Term	Corner frequency	Slope dB/dec	change in Slope db/dec
$(j\omega)^2$	-	+40	
$\frac{1}{1+0.2j\omega}$	$\omega c_1 = \frac{1}{0.2} = 5$	-20	$40-20=20$
$\frac{1}{1+0.02j\omega}$	$\omega c_2 = \frac{1}{0.02} = 50$	-20	$20-20=0$

choose a Low frequency ω_l, $\omega_l < \omega_{c_1}$,

high frequency, ω_h, $\omega_h > \omega_{c_2}$.

Let $\omega_l = 0.5$ rad/sec.

$\omega_h = 100$ rad/sec.

$A = |G(j\omega)|$ in dB

calculate A at $\omega_l, \omega_{c_1}, \omega_{c_2}, \omega_h$.

at $\omega = \omega_l = 0.5$; $A = 20 \log |(j\omega)^2| = 20 \log (\omega^2)$

$= 20 \log (0.5)^2 = -12 db$

$\omega = \omega_{c_1} = 5$, $A = 20 \log |(j\omega)^2| = 20 \log (5)^2 = 28 db$

$\omega = \omega_{c_2} = 50$, $A = \left(\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right) +$

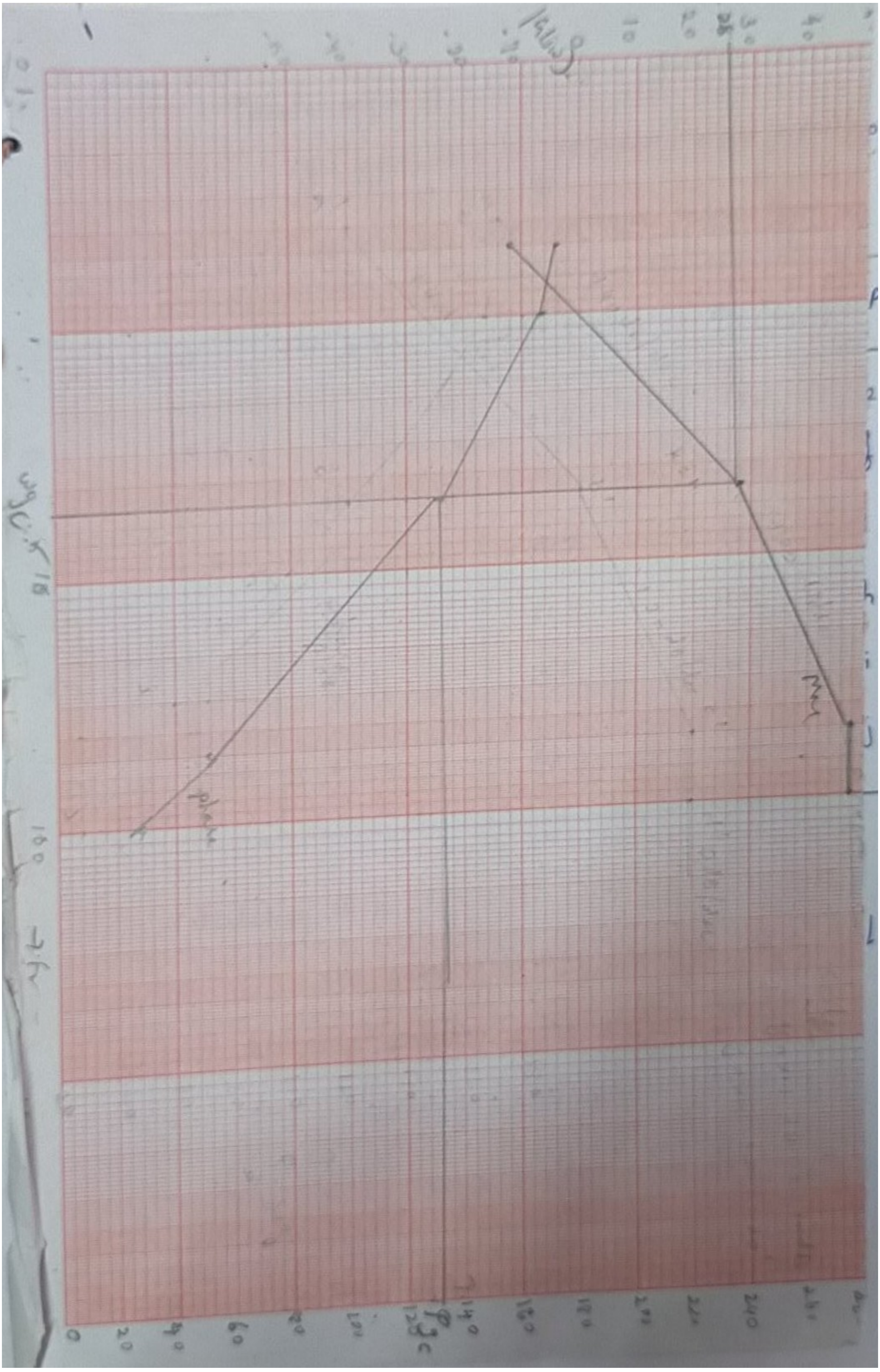
$20 \times \log \frac{50}{5} + 28$ A at $(\omega = \omega_{c_1})$

$= 20 \times \log \frac{50}{5} + 28 = 48 dB$

$\omega = \omega_h = 100$, $A = \left(\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}}\right) +$

A $(\text{at } \omega = \omega_{c_2})$

$= 0 \times \log \frac{100}{50} + 48 = 48 db$



phase plot :

phase angle of $G(j\omega)$ as a function of ω

$$\phi = \angle G(j\omega) = \underbrace{180' - \tan^{-1} 0.2\omega - \tan^{-1} 0.02\omega}_{\phi}$$

phase angle of $G(j\omega)$ for various values of ω.

ω	$\tan^{-1} 0.2\omega$	$\tan^{-1} 0.02\omega$	$\phi = \angle G(j\omega)$	point
0.5	5.7	0.6	174.	e
1	11.3	1.1	168	f
5	45	5.7	130	g
10	63.4	11.3	106	h
50	84.3	45	50	i
100	87.1	63.4	30	j

Calculation of k.

the gain cross over frequency is 5 rad/sec.

At $\omega = 5$ rad/sec, the gain is 28 db

$$20 \log k = -28 \text{ db.}$$

$$\log k = -\frac{28}{20} = 10^{-28/20} = 0.0398$$

$$\boxed{k = 0.0398}$$

polar plot

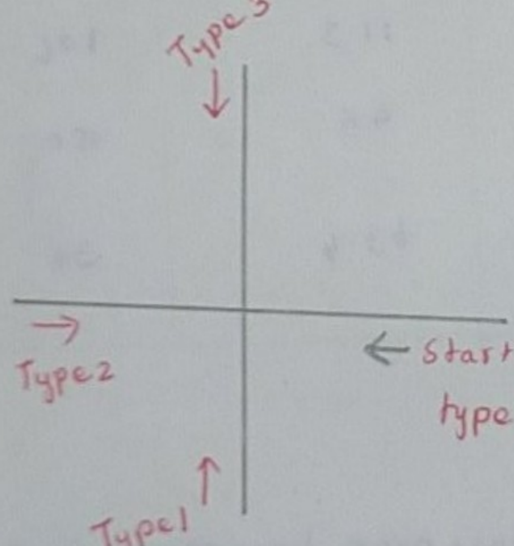
The polar plot of sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ Vs $\angle G(j\omega)$ on polar coordinates as $\omega \to 0 - \infty$. The polar plot is also called as Nyquist plot.

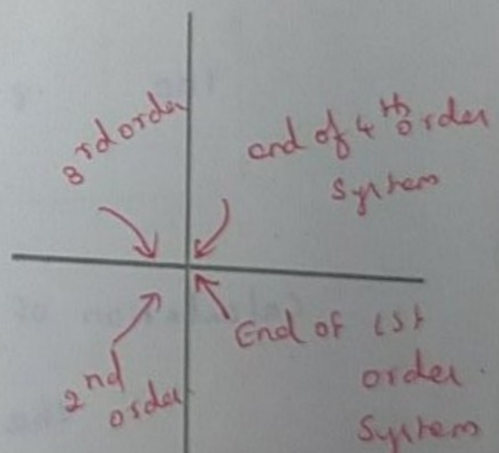
circle → Magnitude.

radial Line → phase angle.

positive phase angle → anticlockwise. from reference 0°

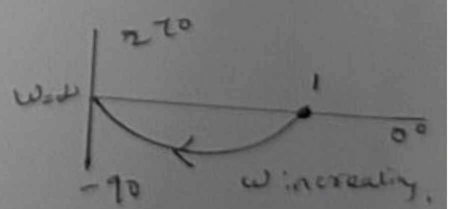
Negative phase angle → clockwise from reference 0°





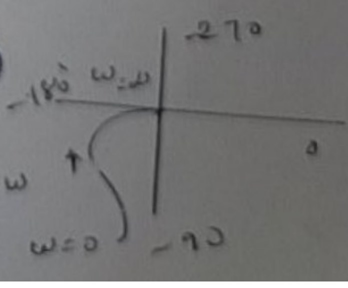
Typical sketches of polar plot.

Type 0 order 1 $\quad G(s) = \dfrac{1}{(1+sT)}$

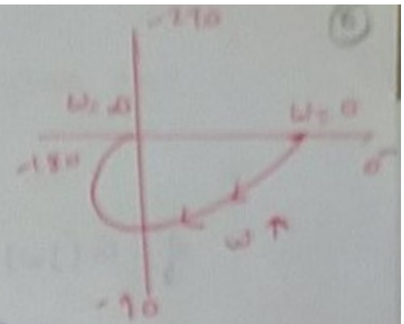


Type 1, order 2

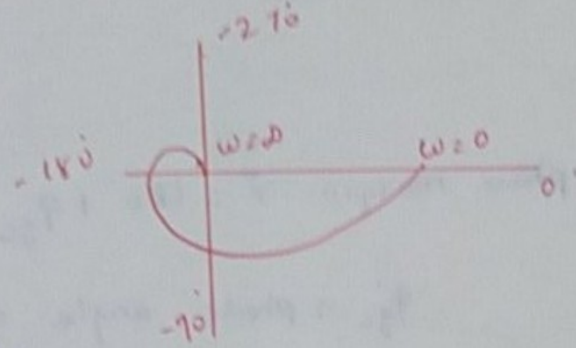
$G(s) = \dfrac{1}{s(1+sT)}$



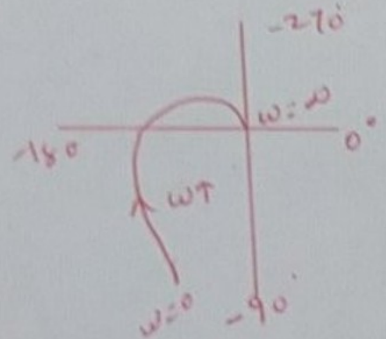
Type 0, order 2 $\quad G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$



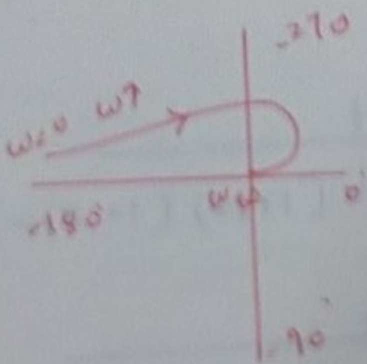
Type 0, order 3 $\quad G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$



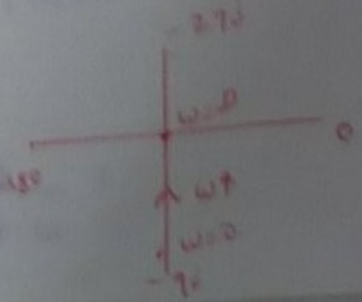
Type 1, order 3 $\quad G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)}$



Type 2, order 4 $\quad G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$



Type 1, order 1 $\quad G(s) = \dfrac{1}{s}$



Gain margin:-

It is defined as the inverse of magnitude of $G(j\omega)$ at phase crossover frequency.

Phase crossover frequency is the frequency at which the phase of $G(j\omega)$ is $180°$

phase margin:

phase margin $\gamma = 180° + \phi_{gc}$

$\phi_{gc} \rightarrow$ phase angle of $G(j\omega)$ at gain crossover frequency.

problem: The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)(1+2s)}$ Sketch the polar plot & determine the gain margin & phase margin

Soln:

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

order 3 type 1 $(a+jb)$ $\sqrt{a^2+b^2}$ $\tan^{-1}\frac{b}{a}$

put $s = j\omega$

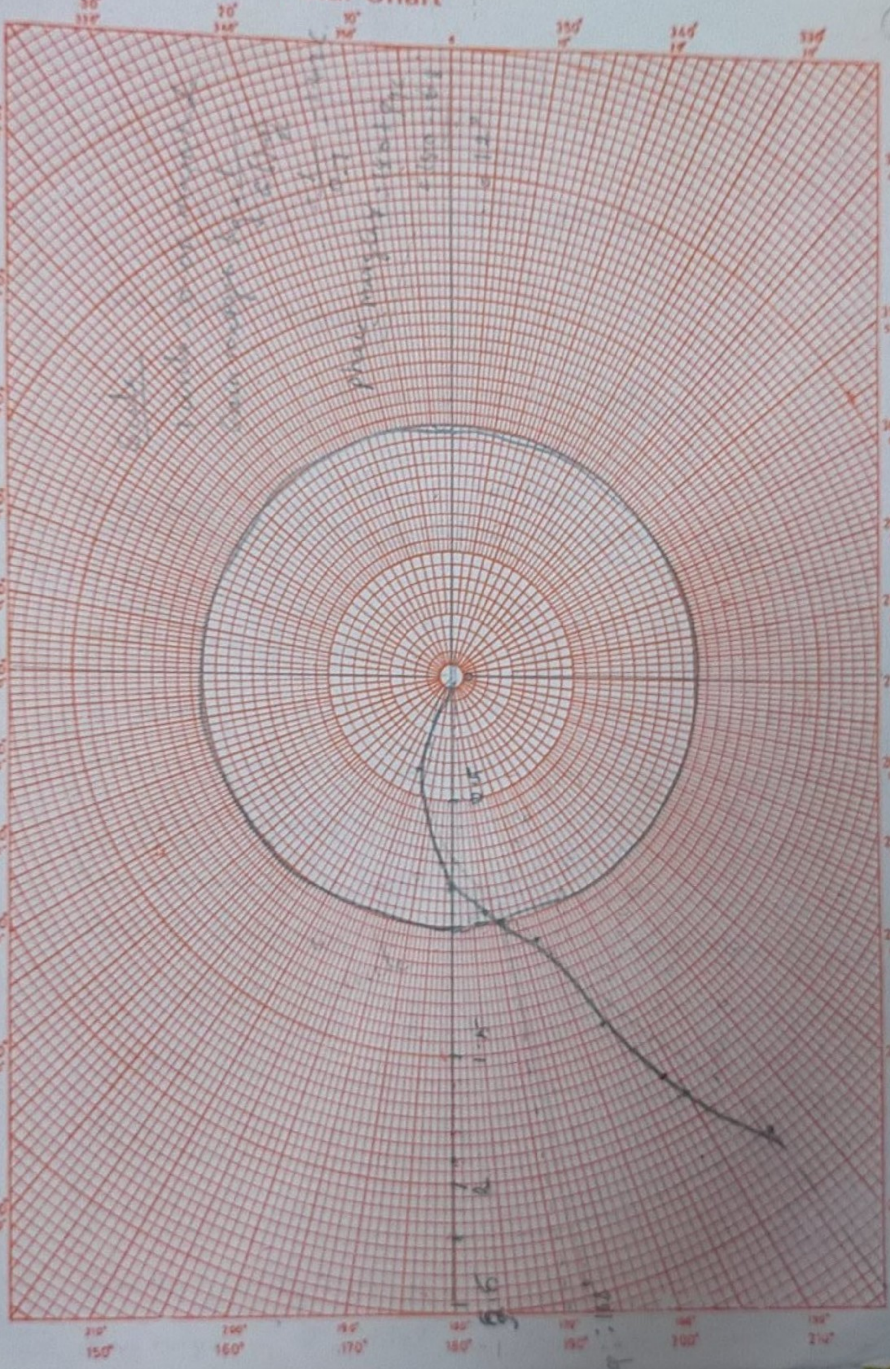
$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

$\tan^{-1}\frac{\omega}{0}$ $|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1^2+\omega^2}\sqrt{1^2+(2\omega)^2}}$

$$G(j\omega) = \frac{1}{\omega\angle 90° \sqrt{1^2+\omega^2}\angle\tan^{-1}\omega \sqrt{1^2+4\omega^2}\angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} \angle -90° - \tan^{-1}\omega - \tan^{-1}2\omega.$$

Polar Chart



$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}}$$

$$= \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

The corner frequency. $\omega_l = 0.1$ $\omega_l, \omega_{c_1}, \omega_{c_2}$ ω_h

$$\omega_{c_1} = \frac{1}{2} = 0.5 \text{ rad/sec}$$

$$\omega_{c_2} = 1 \text{ rad/sec} \qquad \omega_h = 1$$

Magnitude & phase of $G(j\omega)$ at various frequency.

ω rad/sec.	0.35	0.4	0.45	0.5	0.6	0.7	1
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$.	-144	-150	-156	-162	-171	-180	-198

Real & imaginary part of $G(j\omega)$ at various frequency.

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.87	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

Gain Margin $k_g = 1.4286$

phase Margin $\gamma = +12^\circ$.

Determination of closed Loop Response from open loop response.

The closed Loop transfer function of the System is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = M(s).$$

put

$S=j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

Let $M(j\omega) = M\angle\alpha$.

$M \rightarrow$ magnitude of closed Loop T/F.

$\alpha \rightarrow$ phase of closed Loop T/F.

There are two graphical method. to determine the closed Loop frequency from open Loop frequency Response. they are.

1. M & N circle
2. Nichols chat.

M & N circle:

The magnitude of closed Loop transfer function with unity feed back it is in the form of circle for every value of M. These circles are called M circles.

consider the closed Loop transfer function of unity feedback

$$M(s) = \frac{G(s)}{1+G(s)}$$

put $s = j\omega$

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}$$

$G(j\omega) = x + jy$.

x ↓ Real part. jy → imaginary part

$$M(j\omega) = \frac{x+jy}{1+x+jy} = \frac{\sqrt{x^2+y^2}\angle\tan^{-1}\frac{y}{x}}{\sqrt{(1+x)^2+y^2}\angle\tan^{-1}\frac{y}{(1+x)}}$$

$$= \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \angle \tan^{-1}\frac{y}{x} - \tan^{-1}\left(\frac{y}{1+x}\right)$$

M = Magnitude of $M(j\omega)$

$$M = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}.$$

on squaring above equation

$$M^2 = \frac{x^2+y^2}{(1+x^2)^2+y^2}$$

$$x^2(M^2-1) + M^2 2x + M^2 + y^2(M^2-1) = 0.$$

When $M=1$, Straight Line

$$x^2(1-1) + 2x + 1 + y^2(1-1) = 0.$$

$x = -\frac{1}{2}$.

When $M = 1$, represents a straight line passing through

$x = -\frac{1}{2}, y = 0$.

$M \neq 1$, represents a family of circles.

$M \neq 1$, rearranged in the form of equation of circle

$$x^2(M^2-1) + M^2 2x + M^2 + y^2(M^2-1) = 0$$

$\div (M^2-1)$

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2}{M^2-1} + y^2 = 0.$$

add $\frac{M^2}{(M^2-1)^2}$ on both side.

$$\left(x + \frac{M^2}{M^2-1}\right)^2 + y^2 = \frac{M^2}{(M^2-1)^2}.$$

eqn of circle -

$$(x - x_1)^2 + (y - y_1)^2 = r^2.$$

$x_1 = \frac{M^2}{M^2-1}$. When $m = 0$, $x_1 = 0, y_1 = 0$

$m = \infty$, $x_1 = -1, y = 0$.

$y_1 = 0$.

radius $r = \frac{M^2}{M^2-1} = 0$

$M = 0$ the magnitude of circle becomes a point $(0,0)$

$M = \infty$, the magnitude of circle becomes a point $(-1,0)$

When $M<1$, the magnitude of circle to the right of the straight line to $M=1$.

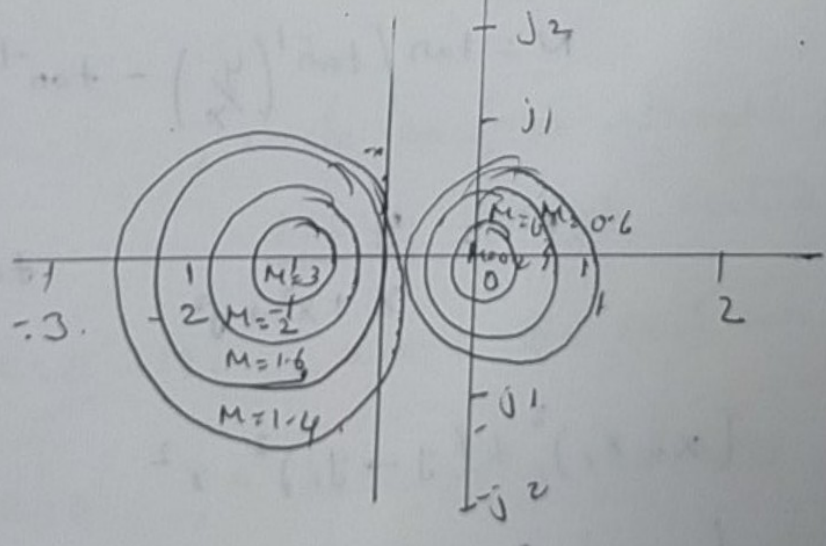
→ passes through $(-\frac{1}{2}, 0)$ & $(0,0)$.

$M\downarrow$, $r\downarrow$, circle becomes a point $(0,0)$.

When $M>1$, → Left of the straight line to $M=1$.

point $(-1,0)$ & $(-\frac{1}{2}, 0)$.

$M\uparrow$, radius $\downarrow$ → circle becomes a point $(-1,0)$ when $M=\infty$.



N-circles:

If the phase of the closed Lopp transfer function with unity feedback is α. $\tan\alpha$ will be in the form for every value of α.

These circles are called N-circles.

consider the closed loop Transfer function.

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}, \quad M(j\omega) = \frac{x+jy}{1+x+jy}.$$

$$= \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \angle \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+y}\right)$$

Let

α - phase of $M(j\omega)$

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\frac{y}{y+x}$$

Let $N = \tan\alpha$.

$$N = \tan\left(\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)\right).$$

$$N = \frac{y}{x+x^2+y^2}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$(x-x_1)^2 + (y-y_1)^2 = r^2.$$

$$\left(x+\frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{2N^2}.$$

$x_1 = \frac{1}{2}$, $y_1 = -\frac{1}{2N}$, $r = \sqrt{\frac{1}{4} + \frac{1}{2N^2}}$.

the equation of N circle should satisfy two point $(0,0)$ & $(-1,0)$.

When.

$x = 0, y = 0$

$$\left(x+\tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2N}\right)^2 = \tfrac{1}{4} + \tfrac{1}{(2N)^2}.$$

$$\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2N}\right)^2 = \tfrac{1}{4} + \tfrac{1}{(2N)^2}$$

$$\tfrac{1}{4} + \tfrac{1}{4N^2} = \tfrac{1}{4} + \tfrac{1}{4N^2}. \rightarrow (1)$$

When $x = -1$ & $y = 0$.

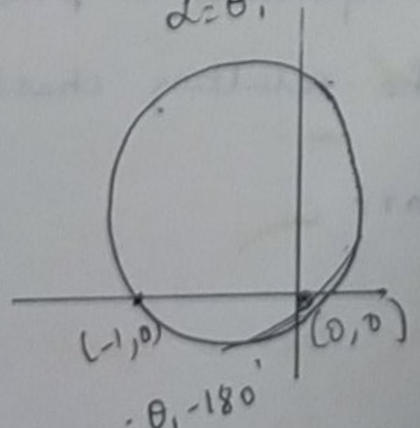
$$\left(x+\tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2N}\right)^2 = \tfrac{1}{4} + \tfrac{1}{(2N)^2}.$$

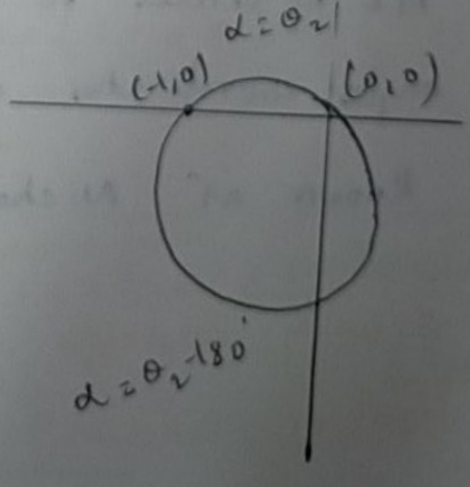
$$\tfrac{1}{4} + \tfrac{1}{4N^2} = \tfrac{1}{4} + \tfrac{1}{4N^2} \rightarrow (2)$$

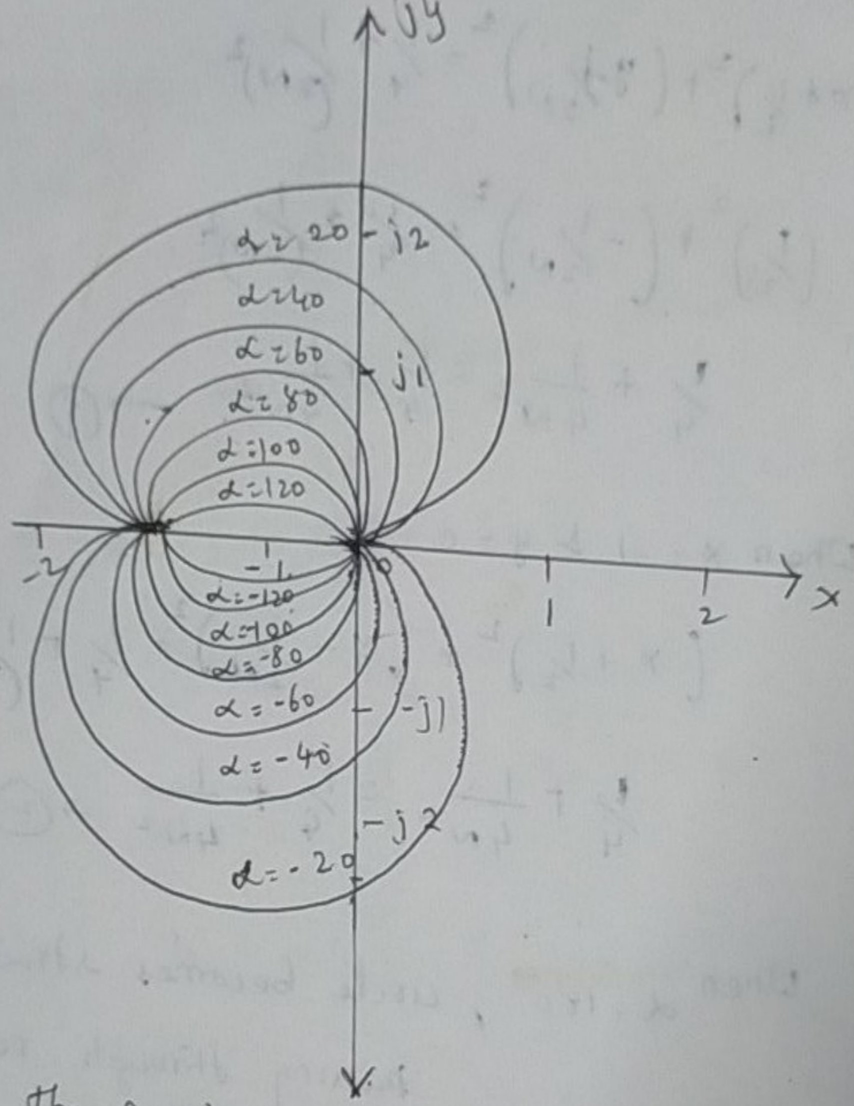
When $\alpha = 180^\circ$, circle becomes straight line passing through real axis.

$\alpha = 0^\circ - 180^\circ$, above the real axis.

$\alpha = 0^\circ$, below the real axis.







The family of constant N-circles.

Nichols chart:

N.B. Nichols transformed the constant M & N circles to log magnitude & phase angle coordinates & the resulting chart is known as Nichols chart.

problem: A unity feedback system has a open loop transfer function $G(s) = \dfrac{20}{s(s+2)(s+5)}$ using Nichols.

chart determine the closed loop frequency response & estimate M_r, ω_r & ω_b.

Soln:

$$G(s) = \frac{20}{s(s+2)(s+5)}.$$

G(s) is converted to time constant or bode form.

$$G(s) = \frac{20}{s \times 2\left(\frac{s}{2}+1\right) 5\left(\frac{s}{5}+1\right)}$$

$$= \frac{\frac{20}{2 \times 5}}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)}$$

$$= \frac{2}{s(1+0.5s)(1+0.2s)}.$$

put $s = j\omega$

$$G(j\omega) = \frac{2}{j\omega(1+0.5j\omega)(1+j0.2\omega)}$$

$$= \frac{2}{\omega \angle 90^\circ \sqrt{1+0.25\omega^2} \angle \tan^{-1} 0.5\omega \sqrt{1+0.04\omega^2} \angle \tan^{-1} 0.2\omega}.$$

$$|G(j\omega)| = \frac{2}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}}.$$

$$|G(j\omega)|_{dB} = 20 \log \left[\frac{2}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}} \right]$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 0.2\omega.$$

Calculate $|G(j\omega)|$ & $\angle G(j\omega)$.

ω	0.2	0.5	1	2	3	4
$\lvert G(j\omega)\rvert$	20	12	5	-4	-10	-15
$\angle G(j\omega)$	-98	-110	-128	-157	-177	-192

Value of $|G(j\omega)|$ & $\angle G(j\omega)$ from bodeplot.

ω	0.2	0.32	0.1	0.74	1.05	1.5	2	2.6	3.2
$\lvert G(j\omega)\rvert$	20	16	12	8	4	0	-4	-8	-11.5
$\angle G(j\omega)$	-98	-102	-110	-118	-130	-146	-156	-168	-180

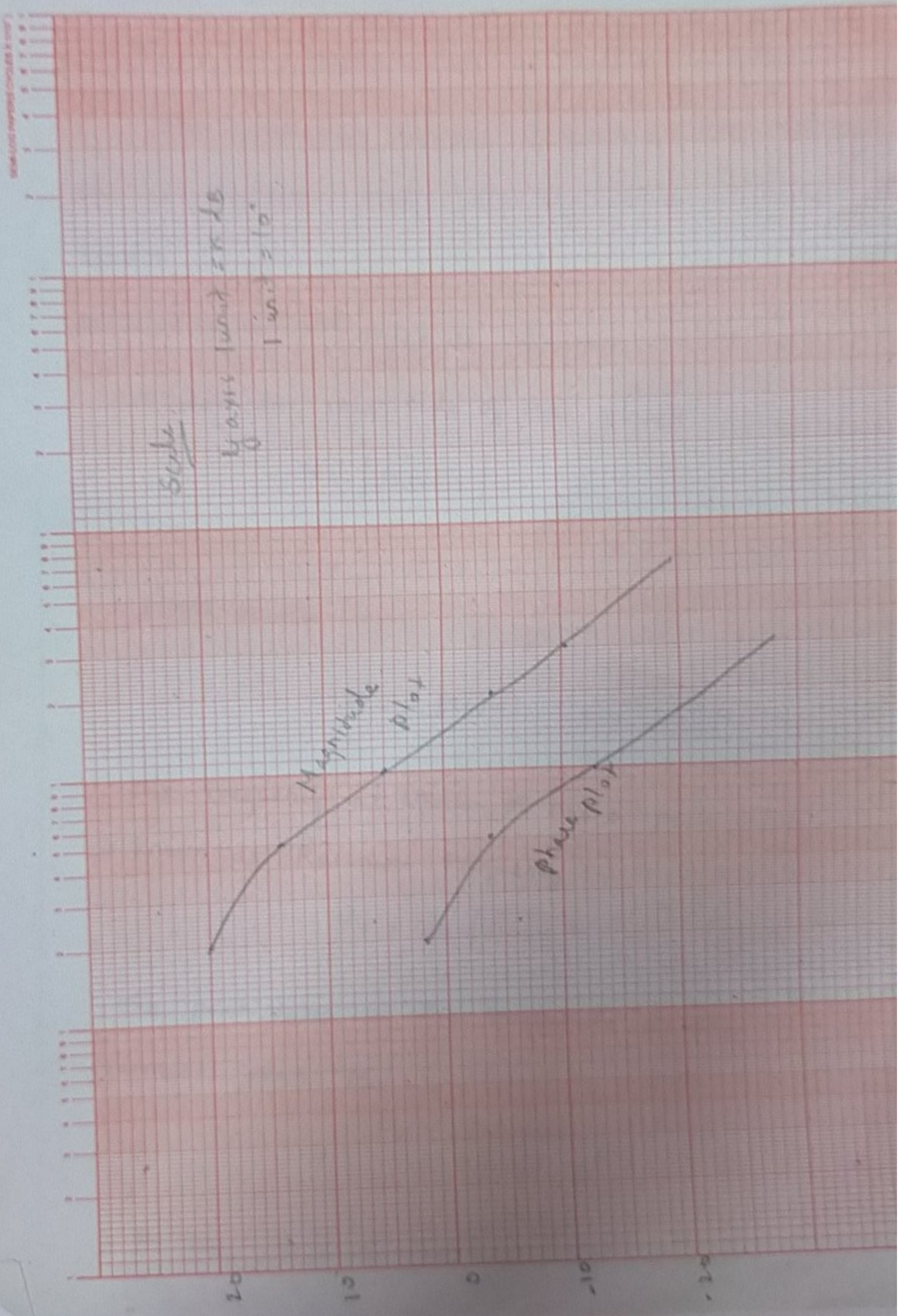
Value of M and α from Nichols chart.

ω	0.36	0.62	1.0	1.2	1.6	1.8	2.	2.1	2.3
M	0.25	1	2	3	4	3	2	1	0.
α	12	21	35	50	78	102	120	130	140.

resonant peak $M_r = +4\,dB$.

$\omega_r = 1.6$ rad/sec

$\omega_b = 2.5$ rad/sec



The freq corresponding to M_r is ω_r, which is the

correlation between frequency domain & time domain Specification

The correlation between frequency & time response has an explicit form only for 1st & 2nd order Systems.

correlation for 2nd order is discussed.

consider the magnitude of the closed Loop system

$$M = |M(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

phase of the closed Loop system.

$$\alpha = \angle M(j\omega) = -\tan^{-1} \frac{2\zeta u}{1-u^2}.$$

The frequency at which M has a peak value is known as Resonant frequency.

$$\omega_r = \omega_n \sqrt{1-2\zeta^2}$$

The peak value of the magnitude is the resonant peak M_r

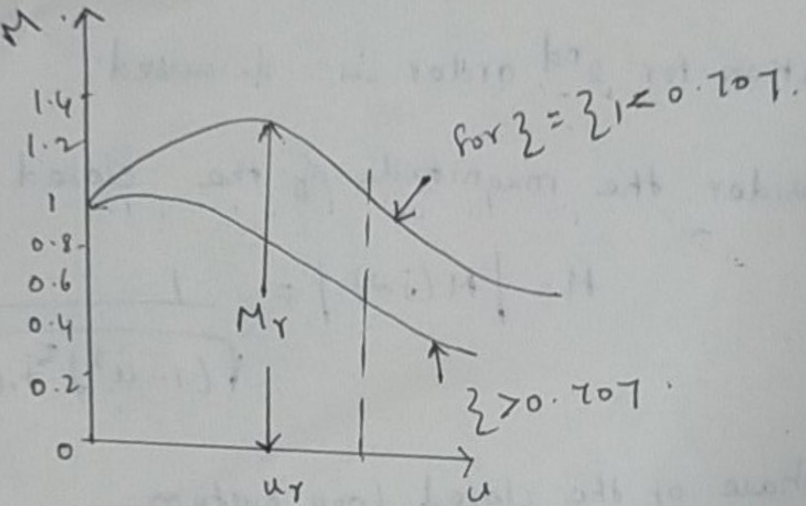
$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

At this freq, slope of the magnitude curve is zero.

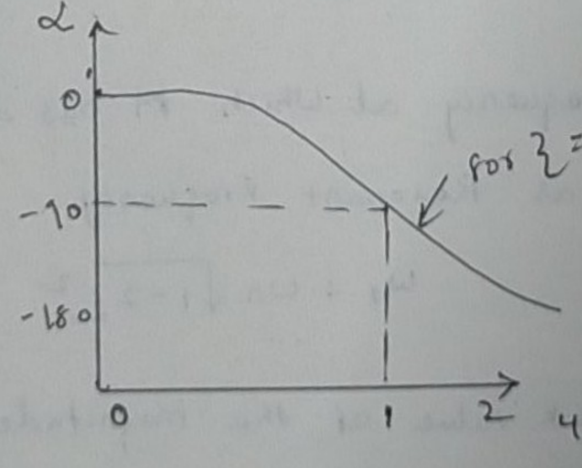
The freq corresponding to M_r is u_r, which is the normalized resonant frequency.

When $\zeta = 0$, $\omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n$

$$\zeta = 0, \quad M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \infty.$$



Magnitude(M) as function of u.



phase α as a function of u.

When $1-2\zeta^2 = 0$, $\omega_r = 0 \rightarrow$ No resonant peak.

$1-2\zeta^2 = 0$, $\zeta^2 = \frac{1}{2}$

$$\zeta = \frac{1}{\sqrt{2}}.$$

For $0 < \zeta < 1/\sqrt{2}$, the resonant freq. Less than ω_n. (4)

resonant peak greater than one

$\zeta > 1/\sqrt{2}$, the condition $\left(\frac{dM}{du}\right) = 0 \rightarrow$ for any real value of ω.

$\zeta > 1/\sqrt{2}$, $M \downarrow$ from $M = 1$ at $u = 0$ & increasing with u. $\rightarrow$ No Resonant peak, $M = 1$.

The frequency at which M has a value of $1/\sqrt{2}$ is called cut off frequency ω_c. The signal frequenc above cut off are greatly attenuated on passing through a system.

For feedback control system,

$M \geq 1/\sqrt{2}$ is defined as the bandwidth of ω_b.

Low pass filter, $\omega_b = \omega_c$

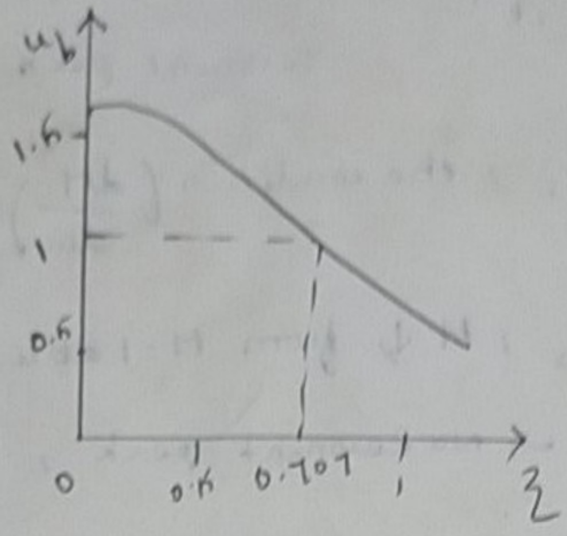
Bandwidth gives a measure of the transien response.

The normalized bandwidth, $u_b = \frac{\omega_b}{\omega_n}$

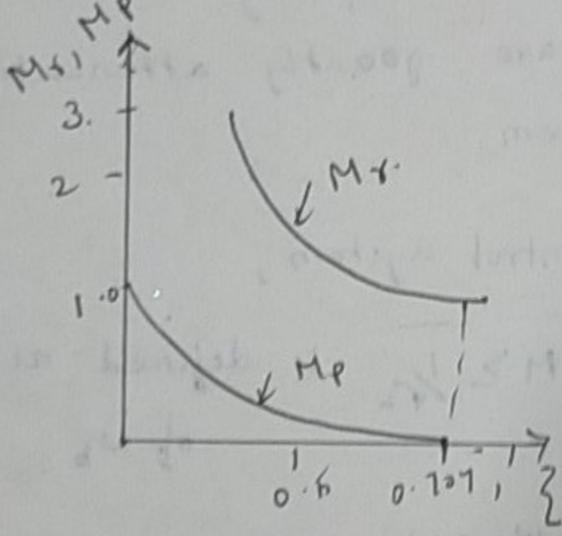
$$= \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}\right]^{1/2}$$

damped frequency of oscillation $\omega_d = \omega_n\sqrt{1-\zeta^2}$

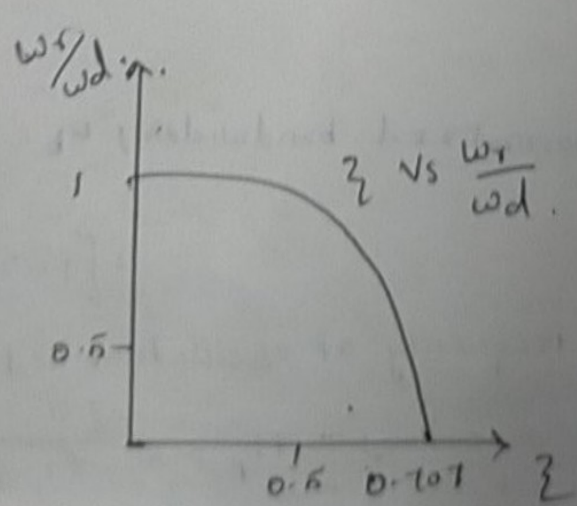
peak overshoot $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$



Normalized bandwidth as a function of ζ



Mr & Mp as a function of ζ



$\frac{\omega_r}{\omega_d}$ as a function of ζ

UNIT - IV

Stability and compensator design

Stability :

Stability refers to the stable working condition of a control system.

- Stable system → o/p is bounded (finite) for any bounded (finite) input.

Asymptotically stable → Absence of input, o/p tends towards zero.

unstable system → if input is finite, o/p is infinite (oscillatory)

Limited stable System → for a bounded i/p signal, o/p has constant amplitude.

∴ System may be stable or unstable for some limited constraints

Absolutely stable system → o/p is stable for all variation of its parameter

conditionally stable system → o/p is stable for limited range of variation of its parameters.

4.1 Characteristic equation:-

The stability of the system depending upon the roots of characteristic equation.

The denominator polynomial of $C(s)/R(s)$ is the characteristic equation.

n^{th} order characteristic equation

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots a_{n-1} s + a_n = 0.$$

the roots of n^{th} order $s = r_1, r_2, r_3 \dots r_n$.

roots are the function of coefficient $a_0, a_1, a_2 \dots a_{n-1}, a_n$.

(i) If all the roots of characteristic equation has negative real parts → System is stable.

(2) If all the roots of the characteristic equation has positive real parts → System is unstable.
(or)
repeated roots on imaginary axis.

(3) condition 1, is satisfied expect for the presence of one or more non repeated roots on the imaginary axis, → system is Limitedly or marginally stable.

Coefficient of characteristic polynomial.

(1) If all the coefficient are positive & if no coefficient is zero, then all the roots are ~~be~~ in the Left half of S-plane.

(2) If any coefficient a_i is equal to zero, then some of the roots may be on imaginary axis. are Right half of S plane.

(3) If any coefficient a_i is negative then atleast one root is in the right half of s-plane.

Necessary condition for stability of the system.

* All the coefficient of its characteristic polynomial be positive.

always true

Roots with negative real part → All coefficient positive.

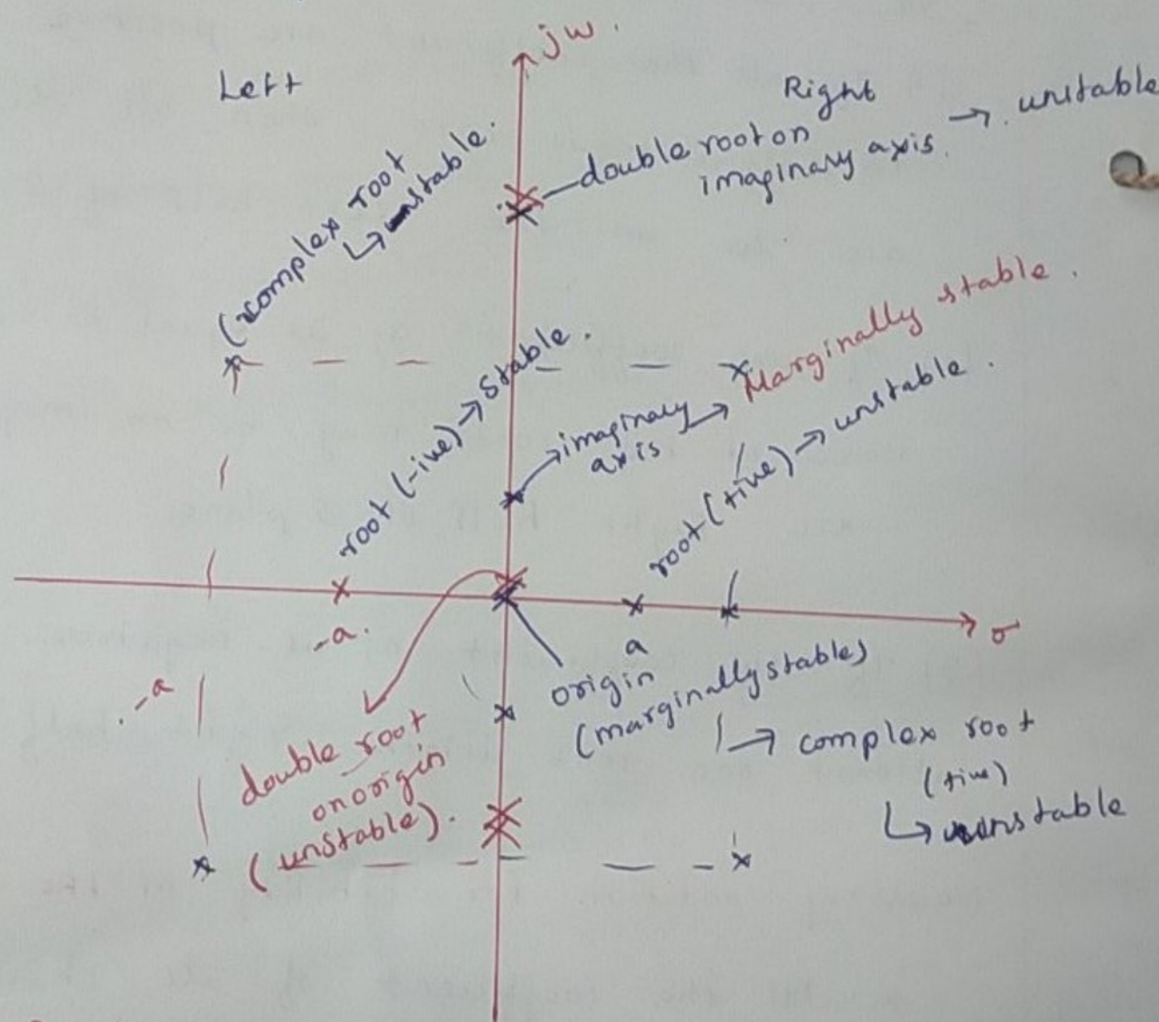
Not always true

All coefficient positive → Roots with negative real part

Roots (Real & negative) → Left half of S plane.
↳ Stable

Roots (Real & positive) → Right half of S-plane
↳ unstable.

Splane



eg. $s^3 + s^2 + 2s + 8 = 0$

$$(s+2)\left(s - \left(\frac{1}{2} - j\frac{\sqrt{15}}{2}\right)\right)\left(s - \left(\frac{1}{2} + j\frac{\sqrt{15}}{2}\right)\right)$$

$$s = -2,\ \frac{1}{2} + j\frac{\sqrt{15}}{2},\ \frac{1}{2} - j\frac{\sqrt{15}}{2}$$

coefficient of the polynomial = +ive.

two Roots are have positive, so Lie on right half of s-plane

$\therefore$ System is unstable.

4.2. Routh Hurwitz Criterion.

Routh hurwitz criterion stability is an analytical procedure for determing whether all the roots of a polynomial have negative real part or not.

Hurwitz criterion → interms of determinants.

Routh criterion → interms of array formulation.

the Routh Stability criterion → based on the ordering the coefficient of characteristic equation.

Necessary & sufficient condition for stability:

→ All the elements in the 1st column of Routh array be positive.

No. of. sign changes in the elements of the 1st column of the Routh array corresponds to the No. of. roots of the characteristic equation in the right half of the S-plane.

1) Construct the Routh array & determine the stability of the system whose characteristic equation is

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Soln:

The characteristic equation of the system.

s^6	1	8	20	16
s^5	2	12	16	
$\div 2$				
s^5	1	6	8	
s^4	1	6	8	
s^3	0	0		
s^3	1	3		
s^2	3	8		
s^1	0.33			
s^0	8			

$s^4 : \frac{1\times8-1\times6}{1} \quad \frac{1\times20-8\times1}{1} \quad \frac{1\times16-0\times1}{1}$

$s^4 : 2 \quad 12 \quad 16$

$\div 2$

$s^4 : 1 \quad 6 \quad 8$

$s^3 : \frac{1\times6-1\times6}{1} \quad \frac{1\times8-1\times8}{1}$

$s^3 : 0 \quad 0$

The auxillary equation is

$$A = s^4 + 6s^2 + 8$$

$$\frac{dA}{ds} = 4s^3 + 12s.$$

$s^3 \quad 4 \quad 12$

$\div 4$

$s^3 \quad 1 \quad 3.$

$s^2 : \frac{1\times6-3\times1}{1} \quad \frac{1\times8-0\times1}{1}$

$3 \quad 8.$

$s^1 = \frac{9-8}{3}$

1st column of the routh array → No sign change.

row → zero

∴ System are marginally stable or Limitedly stable.

The auxillary equation.

$$s^4 + 6s^2 + 8 = 0$$

put $s^2 = x$

$$x^2 + 6x + 8 = 0.$$

$$x = -2, -4.$$

roots of $\quad s^2 = -2, -4$

auxillary equation → . $\quad s = \pm j\sqrt{2}, \pm j\sqrt{4}$

$$s = +j\sqrt{2}, -j\sqrt{2}, +j\sqrt{4}, -j\sqrt{4}.$$

Roots are → $j\sqrt{2}, -j\sqrt{2}, j2, -j2$.

∴ Systems are Marginally stable.

2) Determine the stability of the system corresponding to the characteristic equation $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$ comments on the Location of roots using routh array.

The characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

s^5	1	2	3
s^4	1	2	5
s^3	0	-2.	

Replace. $\overset{\infty}{0}$ by ε.

s^3	ε	-2.

ε

s^3: $\frac{1\times2-2\times1}{1}$ $\quad \frac{1\times3-1\times5}{1}$

$\quad 0 \quad\quad -2$

s^2: $\frac{(\varepsilon\times2)-(-2\times1)}{\varepsilon}$ $\quad \frac{\varepsilon\times5-0\times1}{\varepsilon}$

s^2: $\frac{2\varepsilon+2}{\varepsilon}$ $\quad 5$

$$s^2 \quad \frac{2\varepsilon+2}{\varepsilon} \quad 5$$

$$s^1 \quad \frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}.$$

$$s^0 \quad 5$$

Letting $\varepsilon \to 0$.

s^5 : 1 2 3

s^4 : 1 2 5

s^3 : 0 -2

s^2 : ∞ 5

sign change

s^1 : -2

sign change

s^0 : 5

There are two sign changes in the 1st column of routh array. $\therefore$ System is unstable.

Two roots are lying on right half of s-plane & three roots lying on left half of s-plane.

3) Determine the range of k for stability of unity feedback system whose open loop transfer function is $G(s) = \dfrac{k}{s(s+1)(s+2)}$.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{k}{s(s+1)(s+2)}}{1 + \dfrac{k}{s(s+1)(s+2)}}$$

$$= \frac{k}{s(s+1)(s+2) + k}.$$

The characteristic eqn is

$$s(s+1)(s+2) + k = 0.$$

$$s^3 + 3s^2 + 2s + k = 0.$$

$k > 0$

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3} > 0$	
s^0	k	

$s^1 \quad \dfrac{3\times 2 - k\times 1}{3}$

$s^1 : \dfrac{6-k}{3}$

$s^0 : \dfrac{\frac{6-k}{3} \times k - 0\times 3}{(6-k)/3}$

$s^0 : k$

For stability $k > 0$

s^0 row, $k > 0$.

s^1 row, $\dfrac{6-k}{3} > 0$.

$k \to$ less than 6.

$$0 < k < 6$$

4) Using Routh criterion, determine the stability of the system represented by the characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

$$s^4+8s^3+18s^2+16s+5=0.$$

Soln:

s^4	1	18	5
s^3	8	16	
s^4	1	18	5
s^3	1	2	
s^2	16	5	
s^1	1.7		
s^0	5		

$s^2: \dfrac{1\times 18-2\times 1}{1} \quad \dfrac{1\times 5-0\times 1}{1}$

$s^2: 16 \quad 5$

$s^1: \dfrac{16\times 2-5\times 1}{16}$

$s^1: 1.7$

$s^0: \dfrac{1.7\times 5-0\times 16}{1.7}$

$s^0 = 5.$

$\therefore$ System is stable.

All the four roots are lying on the left half of the s-plane.

4.9 Nyquist stability criterion and performance criterion

The Nyquist stability criterion works on the principle of argument.

P → No. of poles.

Z → zero.

No. of encirclement $N = P - Z$.

* If enclosed 's' plane path contain one poles. direction of encirclement is opposite to the 's' plane.

* If enclosed 's' plane path contain only zeros, the direction of encirclement is same direction to the 's' plane.

The principle of argument to the entire right half of the S-plane by selecting it as a closed path. This selected path is called Nyquist Contour path.

Nyquist stability criterion:-

It states that No. of encirclements about the critical point $(1+j0)$ must be equal to the pole of characteristic equation, which is nothing but the poles of the open loop Transfer function in the right half of the S-plane.

Rules for drawing Nyquist plot.

→ Locate the poles & zero of open loop Transfer function $G(s)\,H(s)$ in the s plane

→ draw the polar plot by varying ω from zero to infinity. If pole or zero present at $s=0$ then varying ~~from~~ $\omega \rightarrow (0 - \infty)$

→ mirror image $\omega \rightarrow (-\infty - 0)$.

→ No. of. infinite radius half circle = No. of. poles or zero of origin

infinite radius half circle starts at the mirror image of polar plot & end at the point where polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using Nyquist stability criterion.

If the critical point $(-1+j0)$ lies outside the encirclement, then the closed loop control system → absolutely stable.

1) Draw the Nyquist plot for the system Whose open Loop Transfer function $G(s)H(s) = \frac{k}{s(s+2)(s+10)}$ determine the range of k for Which closed Loop System is stable.

Soln:

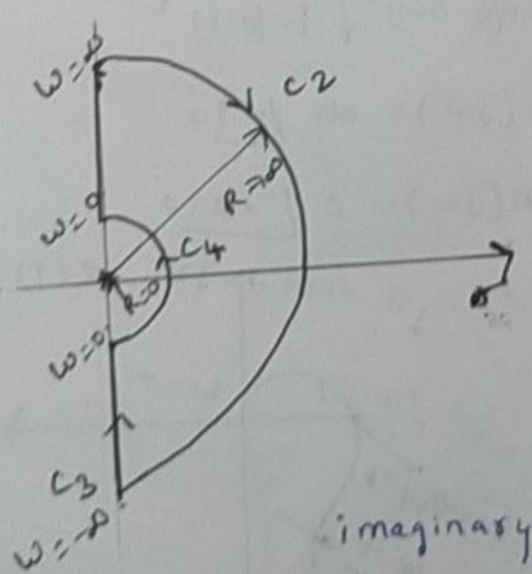
$$G(s)H(s) = \frac{k}{s(s+2)(s+10)}.$$

$$= \frac{k}{20s(1+0.5s)(1+0.1s)}$$

Nyquist contour:

put $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{k}{20j\omega(1+0.5j\omega)(1+0.1j\omega)}$$



$$= \frac{0.05k}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$= \frac{0.05k}{-0.6\omega^2 + j\omega(1-0.05\omega^2)}$$

imaginary part = 0

$$\omega(1-0.05\omega^2) = 0$$

$\omega = \omega_{pc}$.

$$1 - 0.05\omega_{pc}^2 = 0$$

$$\omega_{pc}^2 = \frac{1}{0.05} = 4.47 \text{ rad/sec}$$

$$G(j\omega)H(j\omega) = \frac{+0.05k}{-0.6(4.47)} = -0.0417k.$$

mapping of contour C_1 : $[\omega = 0 \text{ to } \omega = \infty]$

$$G(s)H(s) = \frac{0.05k}{s(1+0.5s)(1+0.1s)}$$

$s = j\omega$

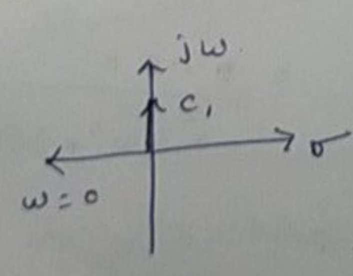
$$G(j\omega)H(j\omega) = \frac{0.05k}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

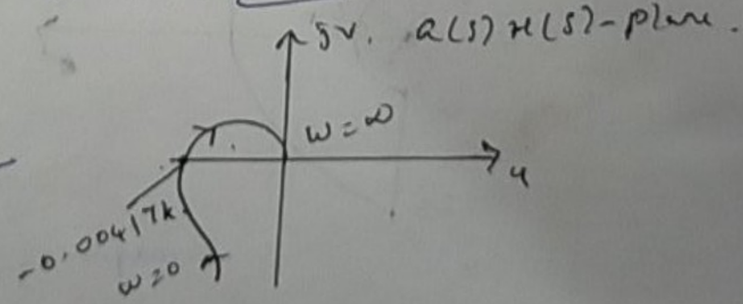
$$= \frac{0.05k}{\omega\angle 90^\circ \sqrt{1+(0.5\omega)^2}\angle \tan^{-1}0.5\omega \sqrt{1+(0.1)^2}\angle \tan^{-1}0.1\omega}$$

$$= \frac{0.05k}{\omega\sqrt{1+(0.5\omega)^2}\sqrt{1+(0.1)^2}} \angle -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.1\omega$$

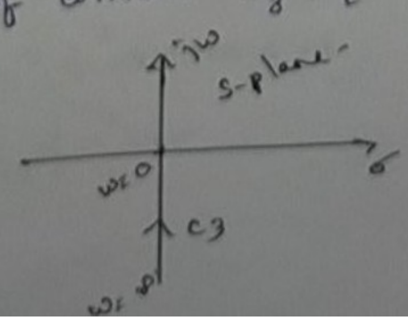
At $\omega = 0$, $G(j\omega)H(j\omega) = \infty \angle -90^\circ$

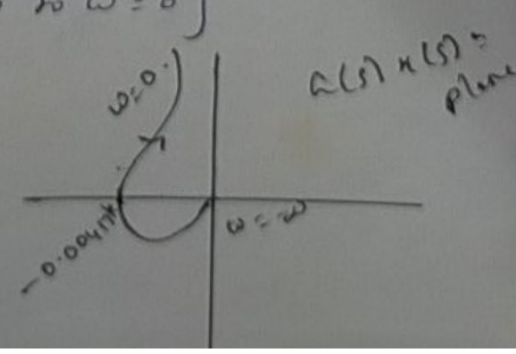
At $\omega = \infty$ $G(j\omega)H(j\omega) = 0 \angle -270^\circ$





mapping of contour C_3 : $[\omega = -\infty \text{ to } \omega = 0]$





mapping of contour c_2 : $\left[\theta = \pi/2 \text{ to } -\pi/2\right]$.

For $R \to \infty \Rightarrow 1+ST = ST$ & $S = \lim_{R\to\infty} Re^{j\theta}$

$$G(s)H(s) = \frac{0.05k}{S(1+0.5S)(1+0.1S)}$$

$$= \frac{0.05k}{S(0.5S)(0.1S)}$$

$$= \frac{0.05k}{S(0.05S^2)} = \frac{k}{S^3}$$

$$G(s)H(s) = \frac{k}{S^3}$$

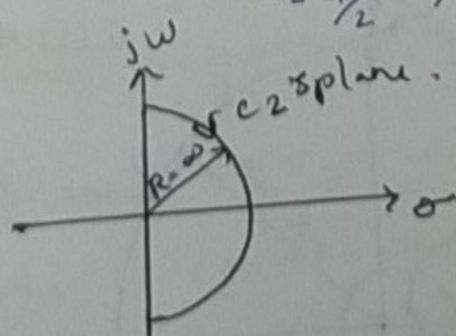
Sub $S = \lim_{R\to\infty} Re^{j\theta}$

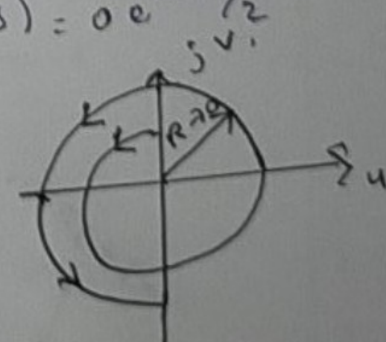
$$G(s)H(s) = \frac{k}{\lim_{R\to\infty} (Re^{j\theta})^3} = \frac{k}{\infty(e^{j\theta})^3}$$

$$= 0\, e^{-j3\theta}$$

When $\theta = \pi/2$, $G(s)H(s) = 0e^{-j3\pi/2}$

$\theta = -\pi/2$, $G(s)H(s) = 0e^{+j3\pi/2}$





mapping of contour c_4 : $\left[\theta = -\pi/2 \text{ to } \pi/2\right]$

For $R \to 0$, $1+ST = 1$, & $S = \lim_{R\to 0} Re^{j\theta}$

$$G(s)H(s) = \frac{0.05K}{s(1+0.5s)(1+0.1s)}$$

$$= \frac{0.05K}{s(1)(1)}$$

$$G(s)H(s) = \frac{0.05K}{s}$$

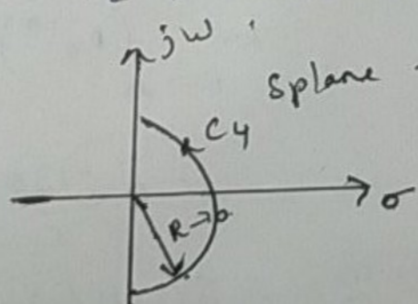
Sub $s = \lim_{R\to 0} Re^{j\theta}$

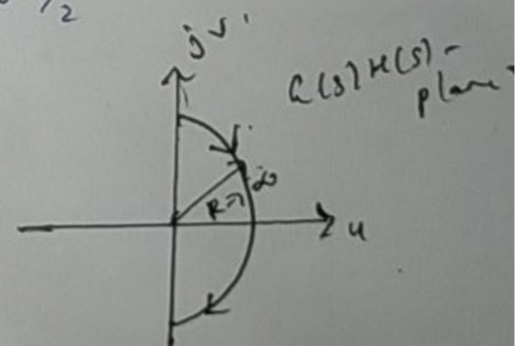
$$G(s)H(s) = \frac{0.05K}{\lim_{R\to 0} Re^{j\theta}} = \frac{0.05K}{0\,e^{j\theta}}$$

$$= \infty e^{-j\theta}$$

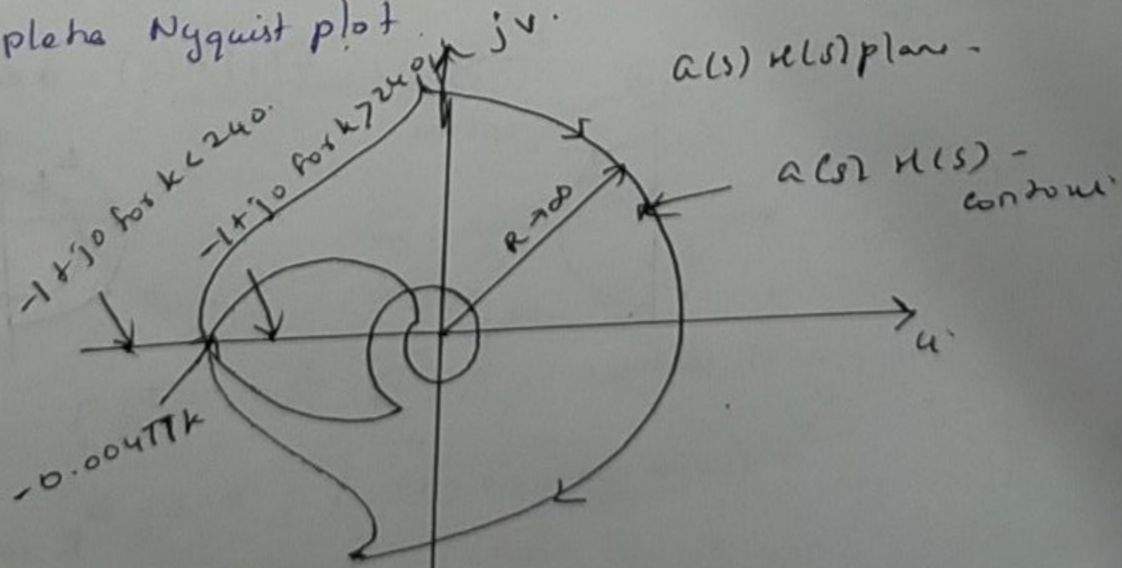
When $\theta = -\pi/2$, $G(s)H(s) = \infty e^{j\pi/2}$

$\theta = \pi/2$, $G(s)H(s) = \infty e^{-j\pi/2}$





Complete Nyquist plot



w= 8'

Limiting value of k : (-1 + j0) point

$$-0.00417k = -1$$

$$k = 240.$$

$k < 240 \left[\begin{array}{l} -0.00417k \Rightarrow k = 230 \Rightarrow -0.96 \\ -0.00417k \Rightarrow k = 220 \Rightarrow -0.92 \end{array} \right]$ Lies Less than -1

* No encirclement

$k > 240 \left[\begin{array}{l} -0.00417k \Rightarrow k = 250 \Rightarrow -1.04 \\ -0.00417k \Rightarrow k = 260 = -1.08 \end{array} \right]$ Lies greater than -1

* There are 2 encirclement in clockwise direction
* Gives no poles on right half of S-plane

Two clockwise encircle.

4.4 <u>Effect of phase Lead network</u>.

1. the velocity constant is usually increased
2. the slope of the magnitude curve is reduced at the gain cross over frequency, with the result relative stability improve.
3. phase margin increased
4. the bandwidth increased
5. the response is faster.

Effect of phase Lag network

1. For a given relative stability, the velocity constant is increased.

2. There is decrease in the gain cross over freq thus decreasing the bandwidth.

3. phase margin increase.

4. Response will be slower

5. Due to decrease in bandwidth, the rise time & the settling time become Large.

Effect of phase Lag-Lead Network.

→ Lead compensator increase bandwidth & speed up the Response & Lag compensation increases the Low frequency gain & improve the Steady State accuracy of the System.

→ Lag-Lead compensator posses two poles & two zeros and so such compensation increases the order of the system by two.

→ cancellation of poles & zero occurs in the compensated system.

4.5 Design of Lag compensator using bode plots.

1. A unity feedback system has an open Loop transfer function $G(s) = \frac{k}{s(1+2s)}$. Design a suitable Lag compensator so that phase margin is 40° & the steady state error for ramp input is Less than or equal to 0.2.

Step 1: Calculate of gain k.

$e_{ss} \le 0.2$ for ramp. $e_{ss} = 0.22$.

$e_{ss} = 1/k_v$ $\quad k_v = \frac{1}{e_{ss}} = \frac{1}{0.2} = 5$.

$$k_v = Lt_{s \to 0}\, s\, G(s) H(s)$$

$$H(s) = 1.$$

$$\therefore k_v = Lt_{s \to 0}\, s \cdot \frac{K}{s(1+2s)} = k$$

$$\therefore k = 5.$$

Step 2: Bode plot of uncompensated system.

$$G(s) = \frac{5}{s(1+2s)}$$

put $s = j\omega$

$$G(j\omega) = \frac{5}{j\omega(1+2j\omega)}$$

Step 3: Magnitude plot.

corner frequency $\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec.

Term	corner freq rad/sec	slope dB/dec	change in slope dB/dec.
$\frac{5}{j\omega}$	—	-20	—
$\frac{1}{1+j2\omega}$	$\omega_c = \frac{1}{2} = 0.5$	-20	$-20-20=-40$

$\omega_l = 0.1$ rad/sec, $\omega_h = 10$ rad/sec.

Let $A = |G(j\omega)|$ in dB.

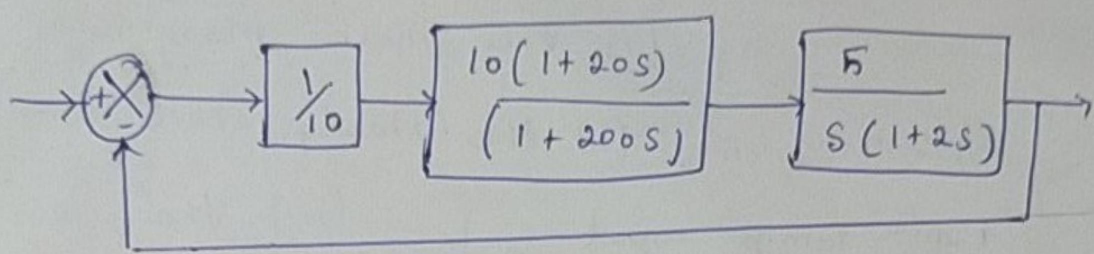
At $\omega = \omega_l$, $A = 20 \log \left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.1} = 34$ dB

$\omega = \omega_c$, $A = 20 \log \left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.5} = 20$ dB

$\omega = \omega_h$, $A = \left[\text{Slope from } \omega_c \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_c}\right] + A(\text{at } \omega = \omega_c)$

$= -40 \times \log \frac{10}{0.5} + 20 = -32$ dB

Step 10: determine the open Loop Transfer function of compensated System.



Open Loop T/f $G_0(s) = \frac{1}{10} \times \frac{10(1+20s)}{(1+200s)} \times \frac{5}{s(1+2s)}$

$$= \frac{5(1+20s)}{s(1+200s)(1+2s)}.$$

put $s = j\omega$

$$G_0(j\omega) = \frac{5(1+j20\omega)}{j\omega(1+200j\omega)(1+2j\omega)}.$$

$$\phi_{gc_0} = \tan^{-1} 20\omega_{gcm} - 90^\circ - \tan^{-1} 200\omega_{gcm} - \tan^{-1} 2\omega_{gcm}$$

$$\phi_{gc_0} = \tan^{-1}(20 \times 0.5) - 90^\circ - \tan^{-1}(200 \times 0.5) - \tan^{-1}(2 \times 0.5)$$

$$= -140^\circ.$$

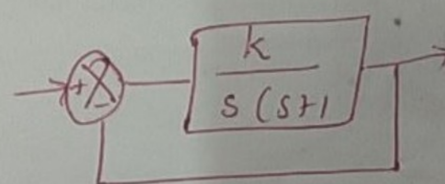
Actual phase margin $\gamma_0 = 180^\circ + \phi_{gc_0}$

$$= 180^\circ - 140^\circ$$

$$= 40^\circ.$$

Design of Lead compensator using Bode plot.

1. Design a phase Lead compensator for the system shown in fig. (i) the phase margin of the system $\geq 45^\circ$. (ii) the steady state error for a unit ramp input $\leq 1/15$. (iii) the gain crossover frequency of the system must be less than 7.5 rad/sec.



Soln: step 1: determine k.

$$e_{ss} \leq 1/15$$

$$e_{ss} = 1/k_v = 1/15 \qquad k_v = 15$$

$$k_v = \lim_{s \to 0} s\,G(s)\,H(s).$$

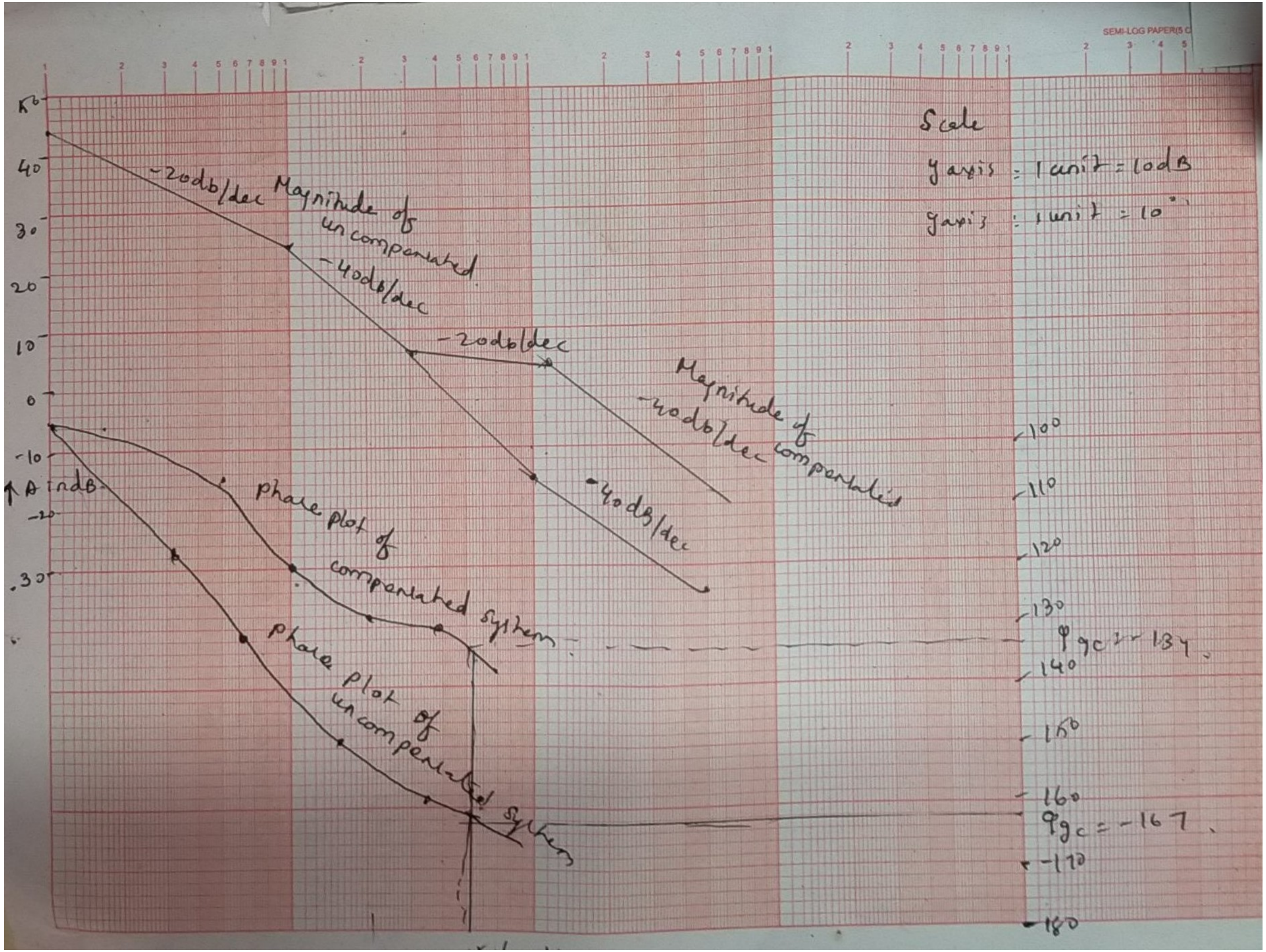
$$G(s) = \frac{k}{s(s+1)} \qquad H(s) = 1$$

$$k_v = k$$

$$\therefore \boxed{k = 15}$$

Step 2: draw bode plot.

$$G(s) = \frac{k}{s(s+1)} = \frac{15}{s(s+1)}$$


SEMI-LOG PAPER
Scale
y axis : 1 unit = 10 dB
y axis : 1 unit = 10°
-20db/dec
Magnitude of uncompenstated
-40dB/dec
-20db/dec
Magnitude of compenstated
-40db/dec
-40dB/dec
Phase plot of compenstated system
Phase plot of uncompenstated system
A in dB
50
40
30
20
10
0
-10
-20
-30
100°
110°
120°
130°
140°
150°
160°
170
180
φgc = -137
φgc = -167

put $s = j\omega$,

$$G(j\omega) = \frac{15}{j\omega(1+j\omega)}.$$

magnitude plot :

$$\omega_{c_1} = 1 \text{ rad/sec}.$$

Term	Corner freq	slope	change in slope.
$\frac{15}{j\omega}$	–	-20	–
$\frac{1}{1+j\omega}$	$\omega_{c_1} = 1$	-20	-40,

$\omega_l = 0.1$ rad/sec , $\omega_h = 10$ rad/sec.

$A = |G(j\omega)|$ in dB.

At $\omega = \omega_l = 0.1$, $A = 20\log\left|\frac{15}{j\omega}\right| = 20\log\left(\frac{15}{0.1}\right)$

$= 44\,dB$.

$\omega = \omega_{c_1} = 1$ rad/sec $\quad A = 20\log\left(\frac{15}{j\omega}\right) = 20\log\left(\frac{15}{1}\right)$

$= 24\,dB$

$\omega = \omega_h = 10$ rad/sec , $A = \cancel{20\log}$[slope from ω_{c_1} to $\omega_h \times \log\frac{\omega_h}{\omega_{c_1}}$] $+ A_{(\omega=\omega_c)}$

$= -40 \times \log\left(\frac{10}{1}\right) + 24 = -16\,dB.$

phase plot :

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}\omega .$$

ω	0.1	0.5	1	2	5	10
ϕ	-96	-117	-135	-153	-169	-174

Step 3 : phase margin of uncompensated system.

$$\gamma = 180^\circ + \phi_{gc}$$

$$= 180^\circ - 167 = 13^\circ .$$

$\therefore$ the system requires phase margin of 45°.

Step 4 : $\phi_m = ?$.

$$\gamma_d \geq 45^\circ .$$

phase Lead $\epsilon = 5^\circ$.

$$\phi_m = \gamma_d - \gamma + \epsilon \quad = 45^\circ - 13 + 5^\circ = 37^\circ$$

Step 5 : determine the transfer function of Lead compensator.

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin 37^\circ}{1 + \sin 37^\circ}$$

$$\simeq 0.248 \simeq 0.25$$

$$\omega_m = -20 \log \frac{1}{\sqrt{\alpha}} = -20 \log \frac{1}{\sqrt{0.25}} = -6\,dB.$$

$\therefore$ ω_m from bode plot $= 5.6$ rad/sec

$$T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{5.6\sqrt{0.25}} = 0.357 \simeq 0.36.$$

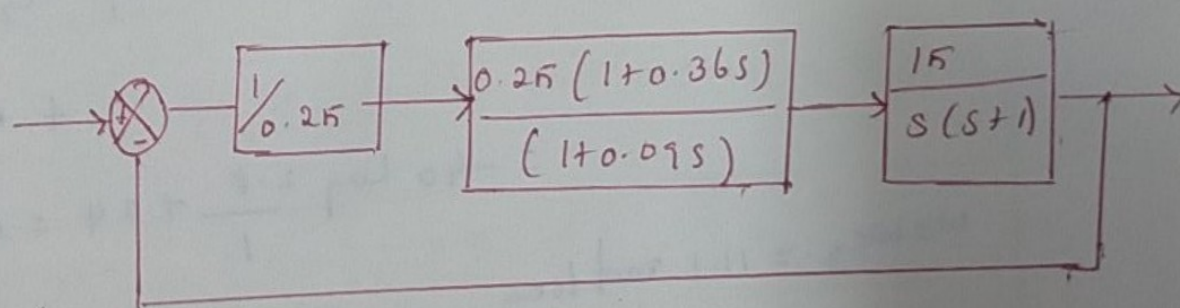
Transfer function of Lead compensator

$$G_c(s) = \frac{s + 1/T}{s + \frac{1}{\alpha T}}$$

$$= \alpha \frac{(1+sT)}{(1+s\alpha T)}$$

$$= 0.25 \frac{(1+0.36s)}{(1+0.09s)}.$$

Step 6: open Loop T/f of compensated system.



$$G_o(s) = \frac{1}{0.25} \times 0.25 \frac{(1+0.36s)}{(1+0.09s)} \times \frac{15}{s(s+1)}.$$

$$= \frac{15(1+0.36s)}{s(1+0.09s)(1+s)}.$$

put $s = j\omega$

$$G_o(j\omega) = \frac{15(1+0.36j\omega)}{j\omega(1+0.09j\omega)(1+j\omega)}.$$

$\omega c_1 = 1$ rad/sec

$\omega c_2 = \frac{1}{0.36} = 2.8$ rad/sec.

$\omega c_3 = \frac{1}{0.09} = 11.1$ rad/sec.

$\omega_l = 0.1$ rad/sec

$\omega_h = 50$ rad/sec.

$\omega = \omega_l, = 0.1$, $A_0 = 20 \text{ Log}\left(\frac{15}{j\omega}\right) = 20 \text{ Log}\left(\frac{15}{0.1}\right)$

$= 44 \text{ dB}$

$\omega = \omega c_1 = 1$, $A_0 = 20 \text{ Log}\left(\frac{15}{j\omega}\right) = 20 \text{ Log}\left(\frac{15}{1}\right) = 24 \text{ dB}$.

$\omega = \omega c_2 = 2.8$, $A_0 = \left[\text{slope from } \omega c_1 \text{ to } \omega c_2 \times \text{Log} \frac{\omega c_2}{\omega c_1}\right]$

$+ A_{\omega = \omega c_1}$.

$= -40 \text{ Log} \frac{2.8}{1} + 24 = 6 \text{ dB}$.

$\omega = \omega c_3 = 11.1$ rad/sec

$A_0 =$ Slope from ωc_2 to $\omega c_3 \times \text{Log} \frac{\omega c_3}{\omega c_2}$

$+ A(\omega = \omega c_2)$.

$= -20 \text{ Log} \frac{11.1}{2.8} + 6 = -6 \text{ dB}$.

$\omega = \omega_h = 50$ rad/sec

$A_0 =$ Slope from ωc_3 to $\omega_h \times \text{Log} \frac{\omega_h}{\omega c_3} +$

$A_{\omega = \omega c_3}$.

$= -40 \text{ Log} \frac{50}{11.1} + (-6) = -32 \text{ dB}$

Term	corner freq	slope dB/dec	Change in slope dB/dec.
$\frac{15}{j\omega}$	—	−20	—
$\frac{1}{1+j\omega}$	$\omega_{c_1} = 1$	−20	$-20-20=-40$
$1+j0.36\omega$	$\omega_{c_2} = \frac{1}{0.36} = 2.8$	20	$-40+20=-20$
$\frac{1}{1+j0.09\omega}$	$\omega_{c_3} = \frac{1}{0.09} = 11.1$	−20	$-20-20=-40$.

phase plot:

$$\varphi = \angle G_o(j\omega) = \tan^{-1} 0.36\omega - 90^\circ - \tan^{-1} 0.09\omega - \tan^{-1}\omega.$$

ω	0.1	0.5	1	2	5	10
φ_o	−94	−109	−120	−128	−132	−142

$$\gamma_o = 180^\circ + \varphi_{gco}$$

$$= 180^\circ - 134^\circ = 46^\circ.$$

Design of Lag Lead compensator using Bode plot

1. Consider the unity feedback system whose open loop transfer function, $G(s) = \frac{k}{s(s+3)(s+6)}$. design a Lag-Lead compensator to meet the following specifications. (i) velocity error constant $k_v = 80$. (ii) phase margin $\gamma \geq 35°$.

Soln:

Step 1: $k = ?$.

velocity error constant $k_v = \underset{s \to 0}{Lt}\ s\,G(s)$.

$$k_v = 80$$

$$\underset{s \to 0}{Lt}\ s\,G(s) = \underset{s \to 0}{Lt}\ s \cdot \frac{k}{s(s+3)(s+6)} = 80.$$

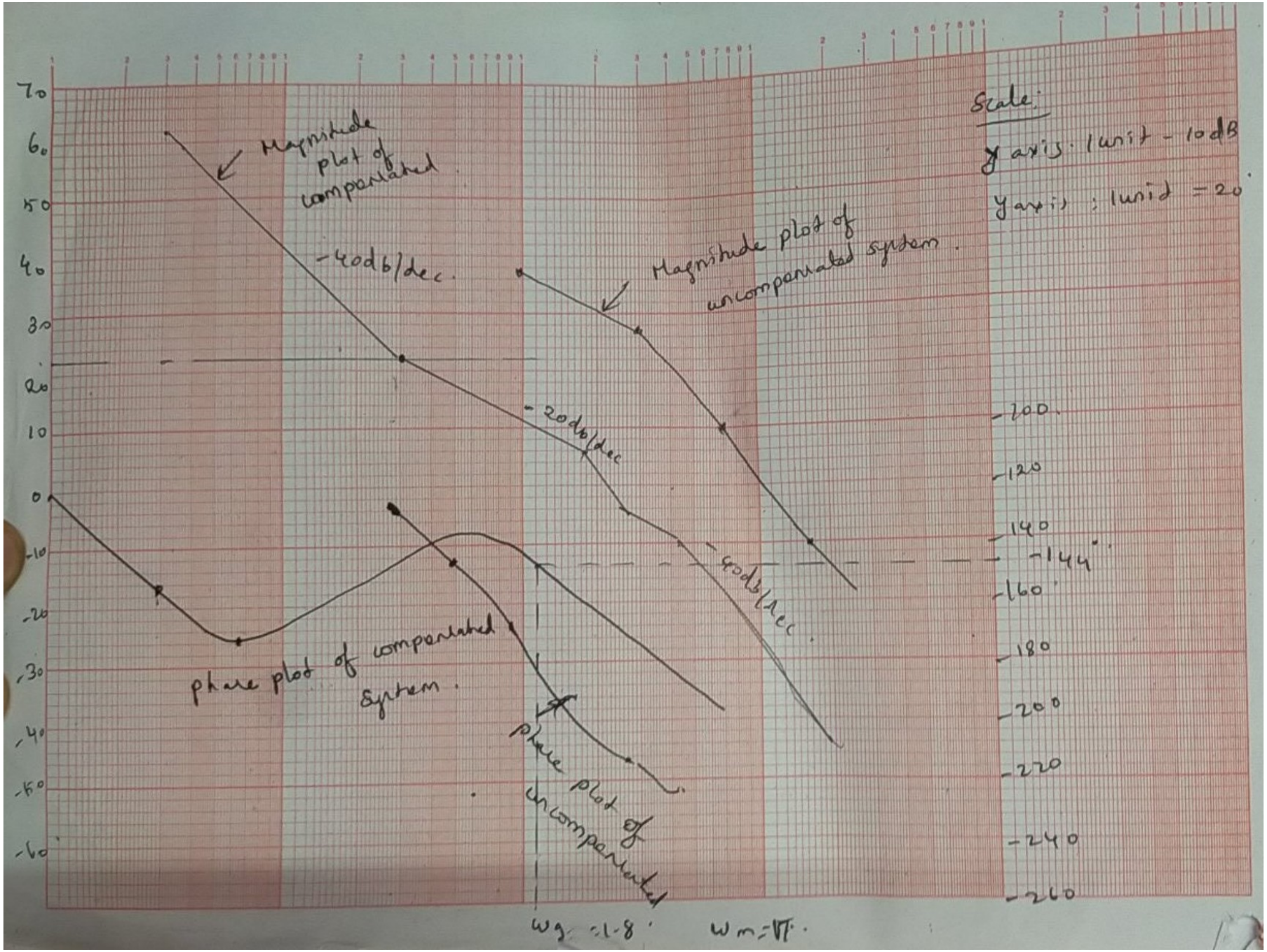
$$\frac{k}{3 \times 6} = 80$$

$$k = 80 \times 3 \times 6 = 1440$$

$$\therefore G(s) = \frac{1440}{s(s+3)(s+6)} = \frac{80}{s(1+0.33s)(1+0.167s)}.$$

Step 2: put $s = j\omega$.

$$G(j\omega) = \frac{80}{j\omega(1+0.33j\omega)(1+0.167j\omega)}.$$



Scale:
y axis: 1unit - 10dB
y axis: 1unit = 20
Magnitude plot of compensated
-40db/dec.
Magnitude plot of uncompensated system.
-20db/dec
-40db/dec.
phase plot of compensated system.
phase plot of uncompensated
wg =1.8
wm=√T.
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-100
-120
-140
-144
-160
-180
-200
-220
-240
-260

Magnitude plot :-

$$\omega_{c_1} = \frac{1}{0.33} = 3 \text{ rad/sec}$$

$$\omega_{c_2} = \frac{1}{0.167} = 6 \text{ rad/sec}$$

Term	Corner freq. rad/sec	Slope dB/dec	Change in slope dB/dec
$\frac{80}{j\omega}$	–	-20	–
$\frac{1}{1+j0.33\omega}$	$\omega_{c_1} = \frac{1}{0.33} = 3$	-20	$-20-20=-40$
$\frac{1}{1+j0.167\omega}$	$\omega_{c_2} = \frac{1}{0.167} = 6$	-20	$-40-20=-60$

$\omega_l = 0.5$ rad/sec , $\omega_h = 20$ rad/sec.

At $\omega = \omega_l$, $A = 20 \log \frac{80}{\omega} = 20 \log \frac{80}{0.5} = 44 \text{ dB}$.

$\omega = \omega_{c_1}$ $= A = 20 \log \frac{80}{\omega} = 20 \log \frac{80}{3} = 28 \text{ dB}$.

$\omega = \omega_{c_2}$, $A = \left[\text{Slope from } \omega_{c_1} - \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] +$ A at $(\omega = \omega_{c_1})$

$$= -40 \times \log \frac{6}{3} + 28 = 16 \text{ dB}.$$

$\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}} \right] +$ A at $(\omega = \omega_{c2})$.

$$= -60 \times \log \frac{20}{6} + 16 = -15 \text{ dB}.$$

phase plot:

$$\varphi = \angle G(j\omega) = -90^{\circ} - \tan^{-1} 0.33\omega - \tan^{-1} 0.167\omega.$$

ω	0.5	1	3	6	10	20
$\angle G(j\omega)$	-104	-118	-160	-198	-222	-244

Step 3:

$\gamma = 180^{\circ} + \varphi_{gc}$ for compensated system.

$\varphi_{gc} = -226^{\circ}$

$\gamma = 180^{\circ} - 226^{\circ} = -46^{\circ}$.

Step 4: choose desired phase margin $\gamma_d = 35^{\circ}$

$\epsilon = 5^{\circ}$.

$\gamma_n = \gamma_d + \epsilon$

$= 35^{\circ} + 5^{\circ} = 40^{\circ}$.

Step 5:

$\gamma_n = 180^{\circ} + \varphi_{gcn}$.

$\varphi_{gcn} = \gamma_n - 180^{\circ} = -140^{\circ}$.

Step 6: $\beta = ?$

$A_{gcl} = 23\,dB$

$A_{gcl} = 20 \log \beta$

$\beta = 10^{A_{gcl}/20} = 10^{23/20} = 14$.

Step 7: determine the transfer function.

zero of Lag compensator $Z_{c_1} = -\frac{1}{T_1} = \frac{\omega_{gcl}}{10}$.

$T_1 = \frac{10}{\omega_{gcl}} = \frac{10}{4} = 2.5$.

pole of Lag compensator $P_{c_1} = \frac{1}{\beta T_1}$

$= \frac{1}{14 \times 2.5}$

$= \frac{1}{35}$.

Transfer function of Lag section $\Big\}\; G_1(s) = \beta \frac{(1+sT_1)}{(1+s\beta T_1)}$

$= 14 \frac{(1+2.5s)}{(1+35s)}$.

Step 8: $\alpha = \frac{1}{\beta} = \frac{1}{14} = 0.07$

dB gain corresponding to $\omega_m = -20 \log \frac{1}{\sqrt{\alpha}}$.

$= -20 \log \frac{1}{\sqrt{0.07}}$

$= -11.5\ db$

$\simeq 12\ dB$.

$\omega_m = 17\ rad/sec$

$$T_2 = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{17\sqrt{0.07}} = 0.22$$

Transfer function of Lead $\Big\}$ $G_2(s) = \alpha\dfrac{(1+sT_2)}{(1+s\alpha T_2)}$

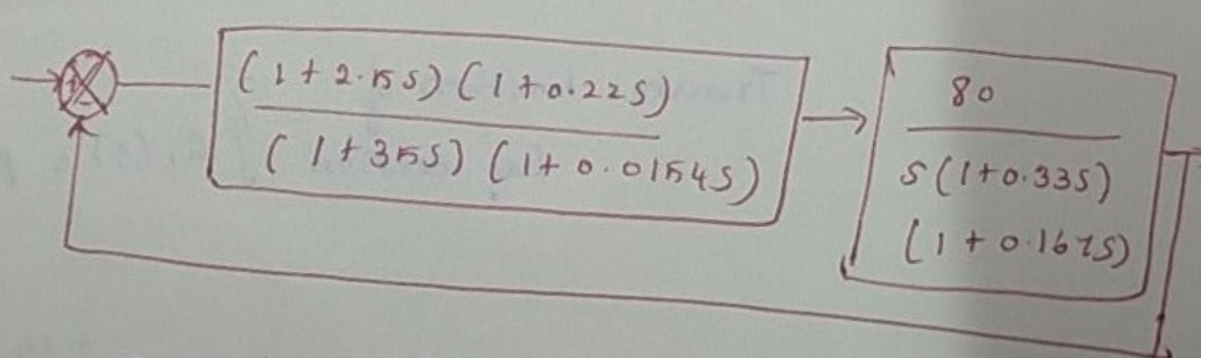
$$= 0.07\frac{(1+0.22s)}{(1+0.0154s)}$$

Step 10: Transfer function of Lag Lead compensator.

$$G_c(s) = G_1(s) \times G_2(s) = 14\frac{(1+2.5s)}{(1+35s)} \times 0.07\frac{(1+0.22s)}{(1+0.0154s)}$$

$$= \frac{(1+2.5s)(1+0.22s)}{(\cancel{1+}s\alpha T_2\;(1+35s)(1+0.0154s)}$$

Step 11: determine open Loop Transfer function.



$$G_0(s) = \frac{80(1+2.5s)(1+0.22s)}{s(1+35s)(1+0.0154s)(1+0.33s)(1+0.167s)}.$$

Step 12: $s = j\omega$

$$G_0(j\omega) = \frac{80(1+j2.5\omega)(1+j0.22\omega)}{j\omega(1+j35\omega)(1+j0.0154\omega)(1+j0.33\omega)(1+j0.167\omega)}$$

Magnitude plot:

$$\omega_{c_1} = \frac{1}{35} = 0.03 \text{ rad/sec}$$

$$\omega_{c_2} = \frac{1}{2.5} = 0.4 \text{ ''}$$

$$\omega_{c_3} = \frac{1}{0.33} = 3 \text{ ''}$$

$$\omega_{c_4} = \frac{1}{0.22} = 4.5 \text{ ''}$$

$$\omega_{c_5} = \frac{1}{0.167} = 6 \text{ rad/sec.}$$

$$\omega_{c_6} = \frac{1}{0.0154} = 65 \text{ rad/sec.}$$

Term	corner freq.	slope	change in slope dB/dec.
$\frac{80}{j\omega}$	—	-20	—
$\frac{1}{1+j3.5\omega}$	$\omega_{c_1} = \frac{1}{3.5} = 0.03$	-20	-40.
$1+j2.5\omega$	$\omega_{c_2} = \frac{1}{2.5} = 0.4$	20	-20
$\frac{1}{1+j0.33\omega}$	$\omega_{c_3} = \frac{1}{0.33} = 3$	-20	-40
$1+j0.22\omega$	$\omega_{c_4} = \frac{1}{0.22} = 4.5$	20	-20
$\frac{1}{1+j0.167\omega}$	$\omega_{c_5} = \frac{1}{0.167} = 6.$	-20	-40
$\frac{1}{1+j0.0154\omega}$.	$\omega_{c_6} = \frac{1}{0.0154} = 65$	-20	-60.

$\omega_l = 0.01$ rad/sec $, \omega_h = 80$ rad/sec.

$\omega = \omega_l \Rightarrow A_0 = 20 \log \dfrac{80}{0.01} = 78$ db

$\omega = \omega_{c1}, \; A_0 = 20 \log \dfrac{80}{0.03} = 68.5$ db $= 68$ db.

$\omega = \omega_{c2}, \; A_0 = -40 \times \log \dfrac{0.4}{0.03} + 68 = 23$ db

$\omega = \omega_{c3}, \; A_0 = -20 \times \log \dfrac{3}{0.4} + 23 = 5$ db.

$\omega = \omega_{c4}, \; A_0 = -40 \times \log \dfrac{4.5}{3} + 5 = -2$ db

$\omega = \omega_{c5}, \; A_0 = -20 \times \log \dfrac{6}{4.5} + (-2) = -4$ db

$\omega = \omega_{c6}, \; A_0 = -40 \times \log \dfrac{65}{6} + (-4) = -45$ db.

$\omega = \omega_h, \; A_0 = -60 \times \log \left(\dfrac{80}{65}\right) + (-45) = -50$ db.

phase plot:

$$\varphi_0 = \angle G_0(j\omega) = \tan^{-1} 2.5\omega + \tan^{-1} 0.22\omega - 90^\circ - \tan^{-1} 35\omega - \tan^{-1} 0.0154\omega - \tan^{-1} \frac{\omega}{0.33} - \tan^{-1} 0.167\omega.$$

ω	0.01	0.03	0.1	0.4	1	4	10	65	80
$\angle G_0(j\omega)$	-108	-132	-152	-138	-126	-144	-168	-220	-228

$\varphi_{gc0} = -144^\circ$

$\gamma_0 = 180^\circ + \varphi_{gc0} = 180^\circ - 144 = 36^\circ$

2) Sketch the Nyquist plot for a system with the open loop Transfer function

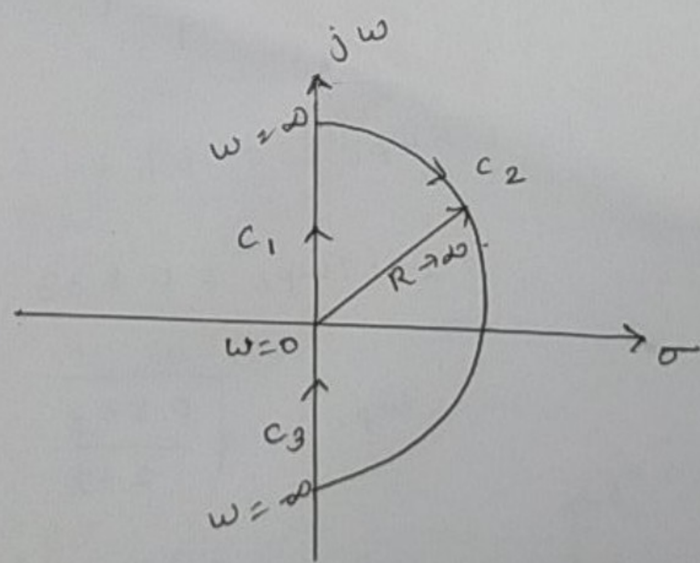
$$G(s)H(s) = \frac{k(1+0.5s)(1+s)}{(1+10s)(s-1)}$$

determine the ranges of value of k for which the system is stable.

Soln:

$$G(s)H(s) = \frac{k(1+0.5s)(1+s)}{(1+10s)(s-1)}.$$

there is no poles. at the origin.



mapping of c_1

$$\omega \rightarrow 0 \rightarrow +\infty.$$

put $s = j\omega$.

$$G(j\omega)H(j\omega) = \frac{k(1+0.5j\omega)(1+j\omega)}{(1+10j\omega)(j\omega-1)}$$

$$= \frac{-k(1-0.5\omega^2)(1+10\omega^2) - 13.5\omega^3 k + j\left[9\omega k(1-0.5\omega^2) - 1.5\omega k(1+10\omega^2)\right]}{(1+10\omega^2)^2 + 9\omega^2}$$

imaginary term is zero.

$$9\omega k(1-0.5\omega^2) - 1.5\omega k(1+10\omega^2) = 0$$

$\omega = \omega_{pc}$

$$9\omega_{pc} k(1-0.5\omega_{pc}^2) = 1.5\omega_{pc} k(1+10\omega_{pc}^2)$$

$$9\omega_{pc} k(1-0.5\omega_{pc}^2) = 1.5\omega_{pc} k(1+10\omega_{pc}^2)$$

$$1-0.5\omega_{pc}^2 = \frac{1.5}{9}(1+10\omega_{pc}^2)$$

$$1-0.5\omega_{pc}^2 = 0.167 + 1.67\omega_{pc}^2$$

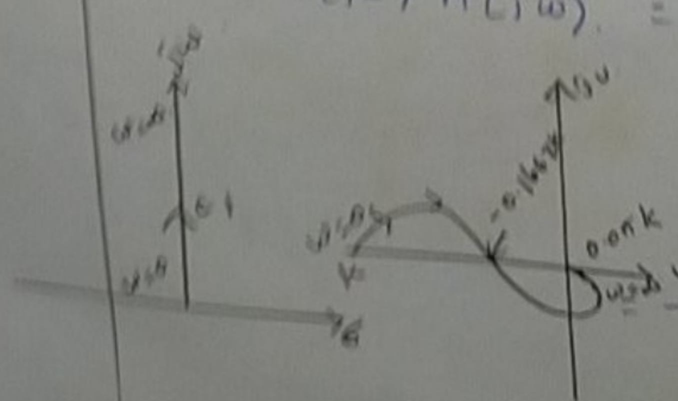
$$2.17\omega_{pc}^2 = 0.833$$

$$\omega_{pc} = \sqrt{\frac{0.833}{2.17}}$$

$$= 0.62 \text{ rad/sec}$$

At $\omega = \omega_{pc} = 0.62$ rad/sec.

$$G(j\omega)H(j\omega) = \frac{-k(1-0.5\omega_{pc}^2)(1+10\omega_{pc}^2) - 13.5\omega_{pc}^2 k}{(1+10\omega_{pc}^2)^2 + (9\omega_{pc})^2}$$



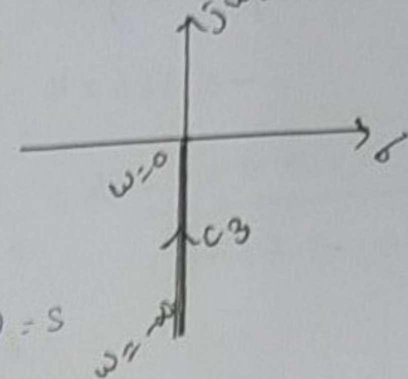
$$= -k\left[\frac{3.913+5.189}{23.464+31.136}\right] = -0.1667k.$$

mapping of Section C_2.

put $s = \lim_{R \to \infty} R e^{j\theta}$

$\theta \to -\pi/2$ to $\pi/2$.

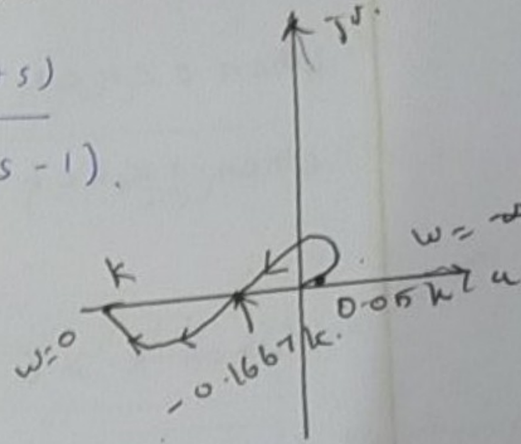
$(1+sT) = sT \qquad (s-1) = s$



$$G(s)H(s) = K\frac{(1+0.5s)(1+s)}{(1+10s)(s-1)}$$

$$= \frac{K\, 0.5s \times s}{10s \times s}$$

$$= 0.05\,K.$$

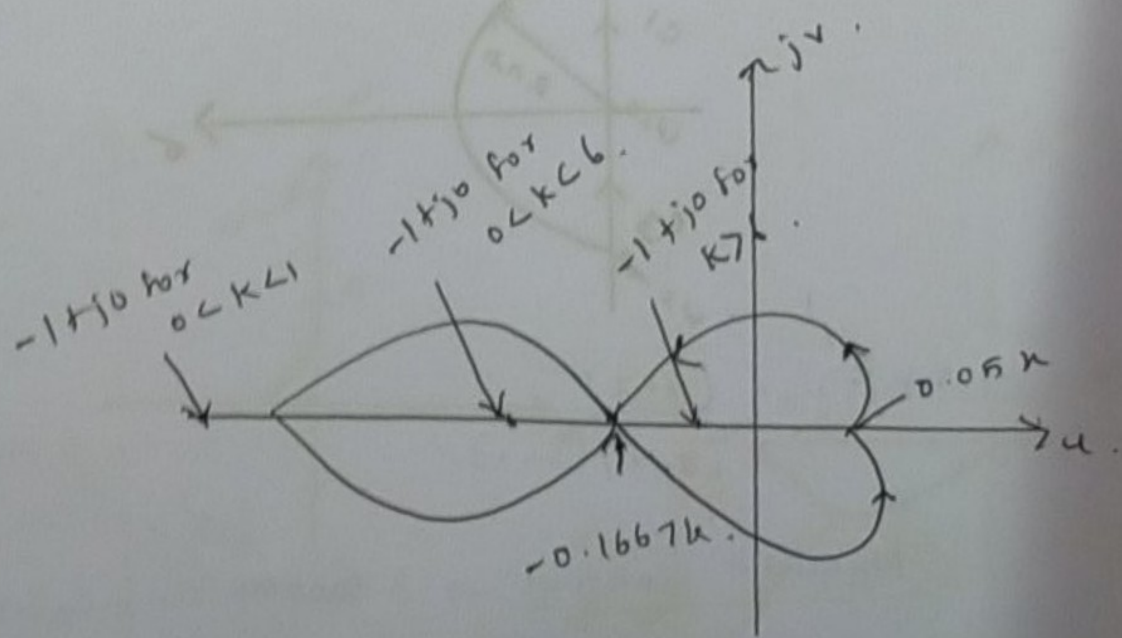


mapping of section C_3

$\omega \to -\infty$ to 0.

inverse polar plot is given by the mirror image of polar plot w.r. to real axis.

Complete Nyquist plot.



Stability Analysis.

$$-0.1667k = -1$$

$$k = \frac{1}{0.1667}$$

$$= 6$$

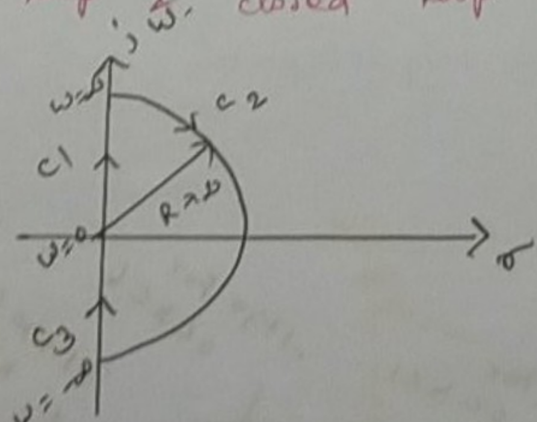
When $0 < k < 1$ → Not encircled, System unstable.

When $1 < k < 6$ → real axis b/w 0 & $-1+j0$. encircled clockwise. unstable.

When $k > 6$ → real axis b/w $-1+j0$ & $-\infty$. encircle anticlockwise, System is stable.

4 By Nyquist stability criterion determine the stability of closed loop system, whose open loop transfer function is given by $G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$. Comment on the stability of open loop & closed loop system.

Soln:



$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

Nyquist contour → 3 section c_1, c_2, c_3

mapping of section C_1

$$\omega \to 0 \to +\infty$$

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)} = \frac{2(1+0.5s)}{(1+s)(-1+s)}.$$

put $s = j\omega$.

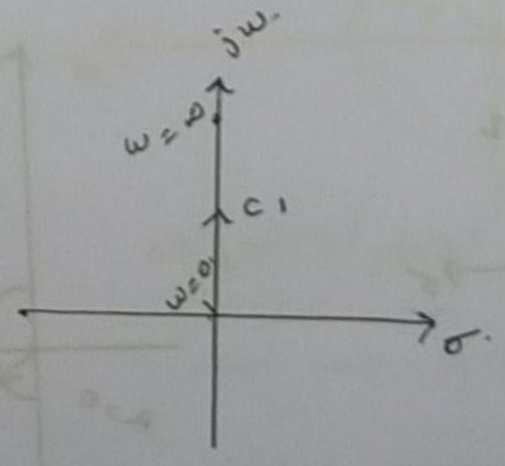
$$G(j\omega)H(j\omega) = \frac{2(1+0.5j\omega)}{(1+j\omega)(-1+j\omega)}.$$

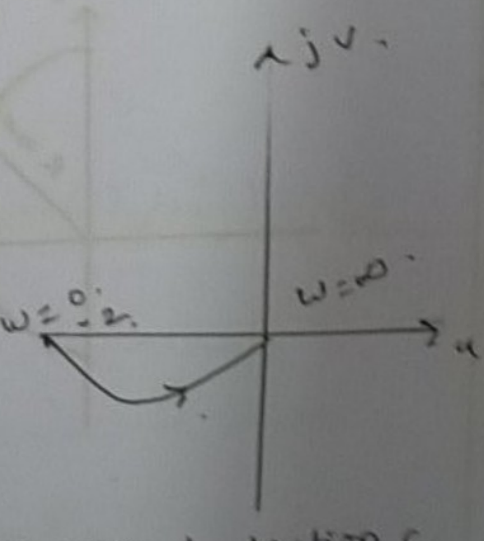
$$= \frac{2\sqrt{1+0.25\omega^2}\ \angle \tan^{-1}0.5\omega}{\sqrt{1+\omega^2}\ \angle \tan^{-1}\omega\ \sqrt{1+\omega^2}\ \angle(180^\circ-\tan^{-1}\omega)}$$

$$= \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2} \angle(-180+\tan^{-1}0.5\omega)$$

$$|G(j\omega)H(j\omega)| = \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}$$

$$\angle G(j\omega)H(j\omega) = -180^\circ + \tan^{-1}0.5\omega$$





Mapping of section C_1

Mapping of section C_2

Put $s = \underset{R\to\infty}{Lt} Re^{j\theta}$

$\theta \to \pi/2$ to $-\pi/2$. $\qquad (1+sT) = sT$.

$$G(s)H(s) = \frac{2(1+0.5s)}{(1+s)(-1+s)}$$

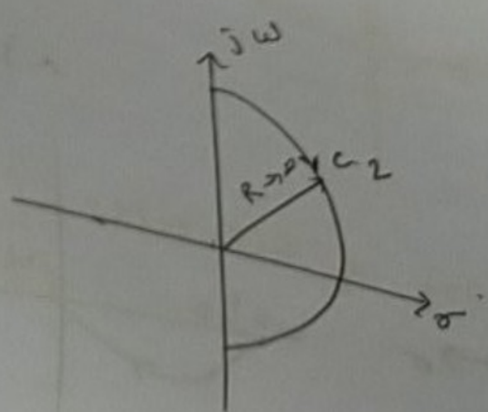
$$= \frac{2\times 0.5s}{s\times s} = \frac{1}{s}.$$

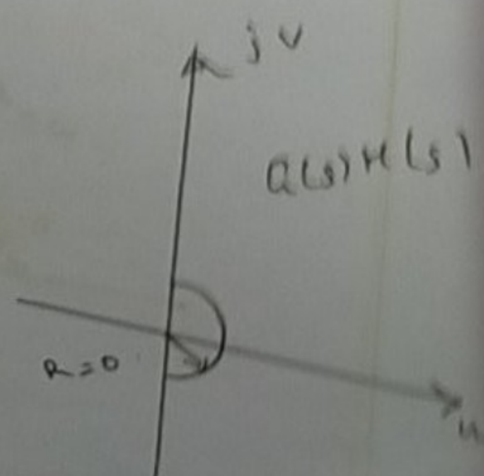
Let $s = \underset{R\to\infty}{Lt} Re^{j\theta}$.

$$= \frac{1}{Lt_{R\to\infty} Re^{j\theta}} = 0e^{-j\theta}.$$

When $\theta = \pi/2$, $G(s)H(s) = 0e^{-j\pi/2}$.

$\theta = -\pi/2$, $G(s)H(s) = 0e^{-j\pi/2}$.



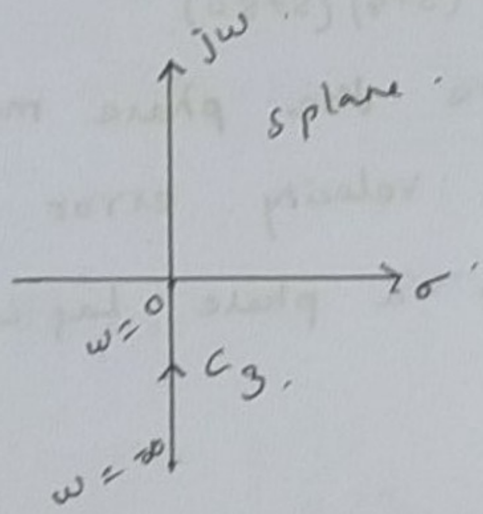


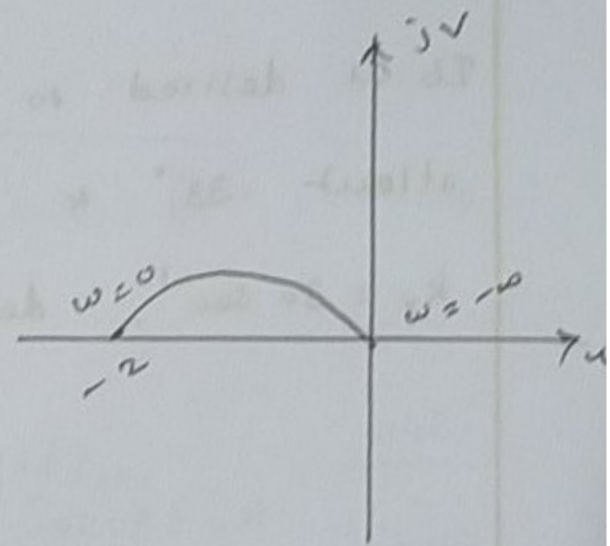
Mapping of section

mapping of section C_3.

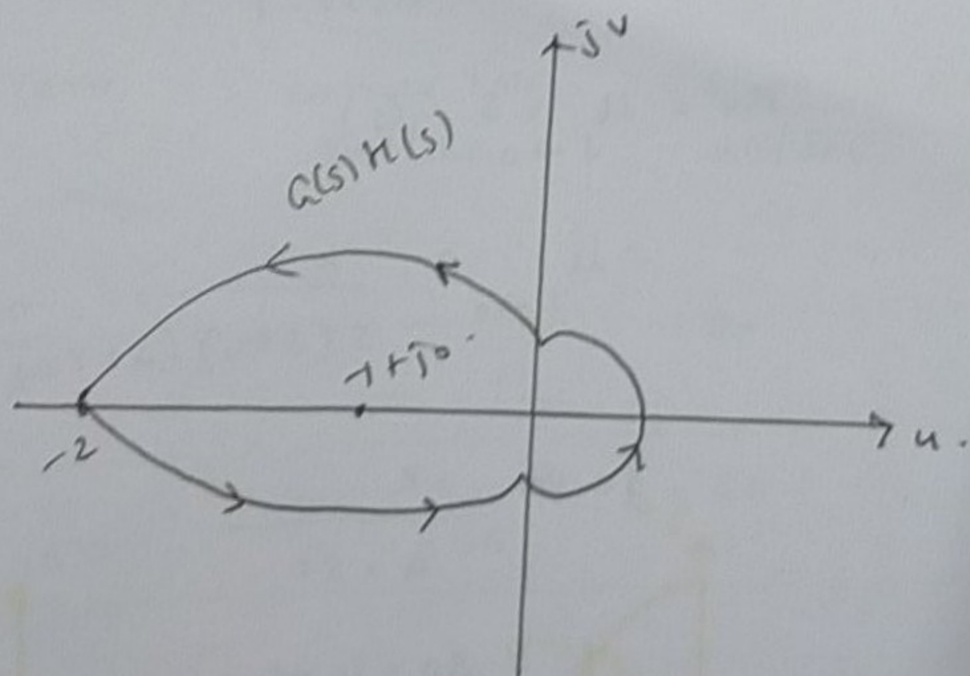
$\omega \rightarrow -\infty$ to 0.

inverse polar plot.





complete Nyquist plot.



Nyquist plot of $G(s)H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$

open loop system is unstable.

closed loop system is stable.

1) The open loop transfer function of certain unity feedback control system is given by

$$G(s) = \frac{k}{s(s+4)(s+80)}.$$

It is desired to have the phase margin to be atleast 33° & the velocity error constant $k_v = 30\ sec^{-1}$. design a phase lag series compen-

Soln:

Step 1: $k_v = 30\ sec^{-1}$

$$k_v = \underset{s \to 0}{Lt}\ s\, G(s) H(s).$$

$$H(s) = 1$$

$$k_v = \underset{s \to 0}{Lt}\ s\, G(s)$$

$$= \underset{s \to 0}{Lt}\ s \cdot \frac{k}{s(s+4)(s+80)}$$

$$30. = \frac{k}{4 \times 80.}$$

$$k = 30 \times 4 \times 80$$

$$\boxed{k = 9600}$$

Step 2:

$$G(s) = \frac{9600}{s(s+4)(s+80)}$$

$$= \frac{30}{S(1+0.25S)(1+0.0125S)}.$$

put $s = j\omega$

$$G(j\omega) = \frac{30}{j\omega(1+j0.25\omega)(1+j0.0125\omega)}.$$

Magnitude plot :-

$$\omega_{c_1} = \frac{1}{0.25} = 4 \text{ rad/sec}$$

$$\omega_{c_2} = \frac{1}{0.0125} = 80 \text{ rad/sec}.$$

Term	corner freq. rad/sec	Slope dB/dec	change in slope dB/dec.
$\frac{30}{j\omega}$	—	-20	—
$\frac{1}{1+j0.25\omega}$	$\omega_{c_1} = \frac{1}{0.25} = 4$	-20	$-20-20=-40$
$\frac{1}{1+j0.0125\omega}$	$\omega_{c_2} = 80$	-20	$-40-20=-60$

$\omega_l = 1$ rad/sec.

$\omega_h = 100$ rad/sec.

At $\omega = \omega_l$, $A = 20\log\left(\frac{30}{j\omega}\right) = 20\log\frac{30}{1} = 30$

$\omega = \omega_{c1}$, $A = 20\log\left(\frac{30}{j\omega}\right) = 20\log\frac{30}{4} = 18$

$\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A$

$= -40\log\frac{80}{4} + 18 = -34$

$\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A$

$= -60 \times \log\frac{100}{80} + (-34)$

$= -40\,dB$.

phase plot:

$\phi = -90^\circ - \tan^{-1} 0.25\omega - \tan^{-1} 0.0125\omega$.

ω	1	4	10	50	80	100
ϕ	-104	-138	-164	-208	-222	-230

Step 3:

$\gamma = 180^\circ + \phi_{gc}$ $\qquad \phi_{gc} = -168$

$= 180 - 168 = 12^\circ$.

The System require 33°. but available is 12°.

Step 4: desired phase margin $\gamma_d = 33^\circ$

$$\gamma_n = \gamma_d + \epsilon$$

initial $\epsilon = 5^\circ$

$$\gamma_n = 33^\circ + 5 = 38^\circ$$

Step 5:

$\omega_{gcn} \Rightarrow$ New gain crossover frequency

ϕ_{gcn} = phase of $G(j\omega)$ at ω_{gcn}

$$\gamma_n = 180^\circ + \phi_{gcn}$$

$$\phi_{gcn} = \gamma_n - 180^\circ$$

$$= 38^\circ - 180 = -142^\circ$$

$\therefore \omega_{gcn} = 4.7$ rad/sec

Step 6: determine β.

from bode plot - ω_{gcn} at 16 dB.

$$A_{gcn} = 20 \log \beta$$

$$A_{gcn} = 16$$

$$\beta = 10^{A_{gcn}/20}$$

$$= 10^{16/20} = 6.3$$

Step 7: determine transfer function of Lag compensator.

$$z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$$

$$= \frac{10}{4.7} = 2.13$$

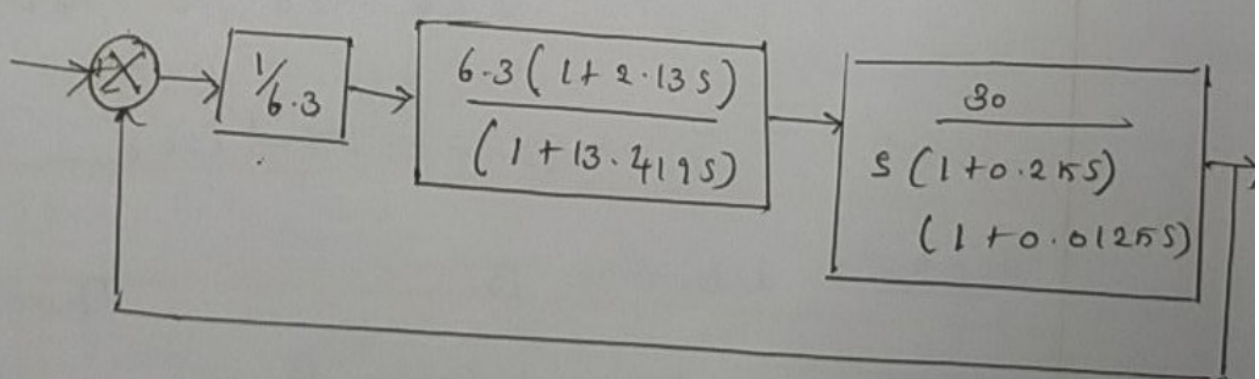
pole of Lag compensator $p_c = \frac{1}{\beta T}$

$$= \frac{1}{6.3 \times 2.13}$$

$$= \frac{1}{13.419}$$

$$G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = 6.3\,\frac{(1 + 2.13s)}{(1 + 13.419s)}$$

Step 8:



$$G_o(s) = \frac{1}{6.3} \times \frac{6.3(1+2.13s)}{(1+13.419s)} \times \frac{30}{s(1+0.25s)(1+0.0125s)}$$

$$= \frac{30(1+2.13s)}{s(1+13.419s)(1+0.25s)(1+0.0125s)}$$

Step 7:

put $s = j\omega$.

$$G_0(j\omega) = \frac{30(1+j2.13\omega)}{j\omega(1+j13.419\omega)(1+j0.25\omega)(1+j0.0125\omega)}$$

$\varphi_{gco}: \ \varphi_0 = \tan^{-1} 2.13\,\omega_{gcn} - 90^\circ - \tan^{-1} 13.419\,\omega_{gcn} - \tan^{-1} 0.25\,\omega_{gcn} - \tan^{-1} 0.0125\,\omega_{gcn}$.

$\varphi_{gco} = \tan^{-1}(2.13 \times 4.7) - 90^\circ - \tan^{-1}(13.419 \times 4.7) - \tan^{-1}(0.25 \times 4.7) - \tan^{-1}(0.0125 \times 4.7)$

$\approx -147^\circ$.

$\gamma_0 = 180^\circ + \varphi_{gc}$.

$= 180^\circ - 147$

$\boxed{\gamma_0 = 33^\circ}$

The actual phase margin of the compensated system satisfy the requirements.

$\therefore$ Design is acceptable.

ii) design a lead compensator for a unity feedback System with open Loop transfer function.

$$G(s) = \frac{k}{s(s+1)(s+5)}$$ to satisfy the following

Specification (i) velocity error constant $k_v \geq 50$.

(ii) phase margin $\geq 20^\circ$.

Soln:

Step 1:

$$k_v \geq 50.$$

$$k_v = \lim_{s \to 0} s\,G(s)$$

$$= \lim_{s \to 0} s \cdot \frac{k}{s(s+1)(s+5)} = \frac{k}{5}$$

$$50 \times 5 = k$$

$$\boxed{k = 250}$$

Step 2: draw bode plot

$$G(s) = \frac{k}{s(s+1)(s+5)}.$$

put $s = j\omega$

$$G(j\omega) = \frac{50}{j\omega(j\omega+1)(1+j0.2\omega)}.$$

Magnitude plot:

$\omega_{c_1} = 1$ rad/sec

$\omega_{c_2} = \frac{1}{0.2} = 5$ rad/sec.

term	corner freq.	Slope db/dec	change in slope
$\frac{50}{j\omega}$	—	-20	—
$\frac{1}{1+j\omega}$	$\omega_{c_1} = 1$	-20	$-20-20=-40$.
$\frac{1}{1+j0.2\omega}$	$\omega_{c_2} = \frac{1}{0.2} = 5$	-20	$-40-20=-60$.

$\omega_l = 0.5$ rad/sec , $\omega_h = 10$ rad/sec.

$A = |G(j\omega)|$ in db.

$\omega = \omega_l$, $A = 20 \log\left(\frac{50}{j\omega}\right) = 20 \log\left(\frac{50}{0.5}\right) = 40$ db

$\omega = \omega_{c_1}$, $A = 20 \log\left(\frac{50}{j\omega}\right) = 20 \log \frac{50}{1} = 34$ db.

$\omega = \omega_{c_2}$, $A =$ slope from ω_{c_1} to $\omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}} + A$

$= 6$ db.

$\omega = \omega_h$, $A = -60 \times \log \frac{10}{5} + 6 = -12$.

phase plot:

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}0.2\omega.$$

ω	0.1	0.5	1	5	10
ϕ	-96°	-122	-146	-214	-238

Step 3:

$\gamma = 180^\circ + \phi_{gc}$ $\qquad$ $\phi_{gc} = -224^\circ$.

$= 180^\circ - 224^\circ = -44^\circ$.

∴ hence Lead compensation have 20°.

Step 4: ϕ_m

$\gamma_d \geq 20^\circ$

$\epsilon = 5^\circ$.

$\phi_m = \gamma_d - \gamma + \epsilon \quad = 20^\circ - (-44^\circ) + 5^\circ = 69^\circ$.

∴ if Lead angle is greater than 60°.

$\phi_m = \dfrac{69^\circ}{2}$ $\qquad$ two Lead compensator is realised

$= 34.5^\circ$.

Step 5:

$$\alpha = \frac{1-\sin\varphi_m}{1+\sin\varphi_m} = \frac{1-\sin 34.5^\circ}{1+\sin 34.5^\circ} = 0.28.$$

$$\omega_m = -20\log\frac{1}{\sqrt{\alpha}} = -20\log\frac{1}{\sqrt{0.28}}$$

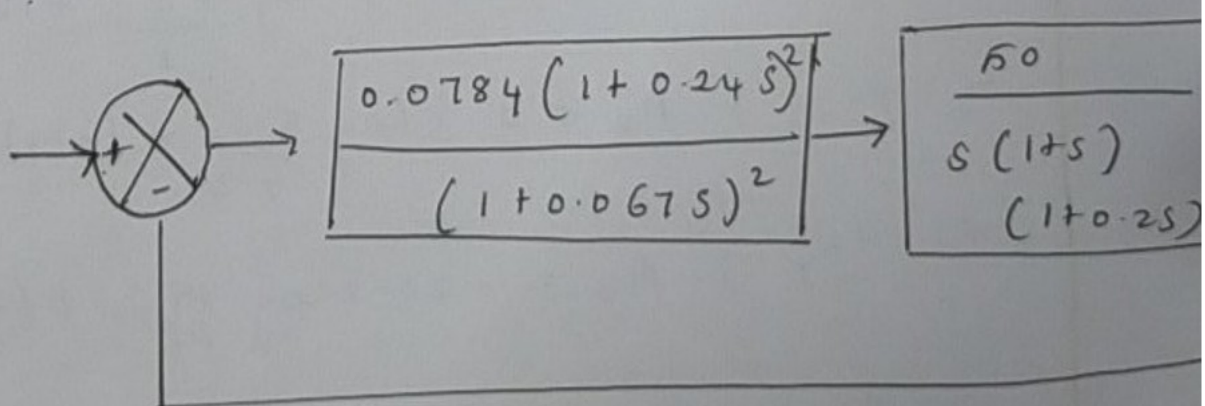
$$= -5.5\ db.$$

$$\omega_m = 7.8\ rad/sec.$$

$$T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{7.8\sqrt{0.28}} = 0.24.$$

$$G_c(s) = \frac{\left(s+1/T\right)^2}{\left(s+\frac{1}{\alpha T}\right)^2} = 0.0784\,\frac{(1+0.24s^2)}{(1+0.067s)^2}$$

Step 6:



$$G_o(s) = \frac{4\left(1+0.24s\right)^2}{s(1+s)(1+0.2s)(1+0.067s)^2}$$

Step 7:

put $s = j\omega$.

$$G_0(j\omega) = \frac{4(1+j0.24\omega)^2}{j\omega(1+j\omega)(1+j0.2\omega)(1+j0.067\omega)^2}$$

Magnitude plot:

$\omega c_1 = 1$, $\omega c_2 = \frac{1}{0.24} = 4.2$, $\omega c_3 = \frac{1}{0.2} = 5$, $\omega c_4 = \frac{1}{0.06}$.

$\omega_l = 0.5$, $\omega_h = 30$ rad/sec.

$A_0 = |G_0(j\omega)|$ in db.

$\omega = \omega_l$, $A_0 = 20\log\left(\frac{4}{j\omega}\right) = 18$ db.

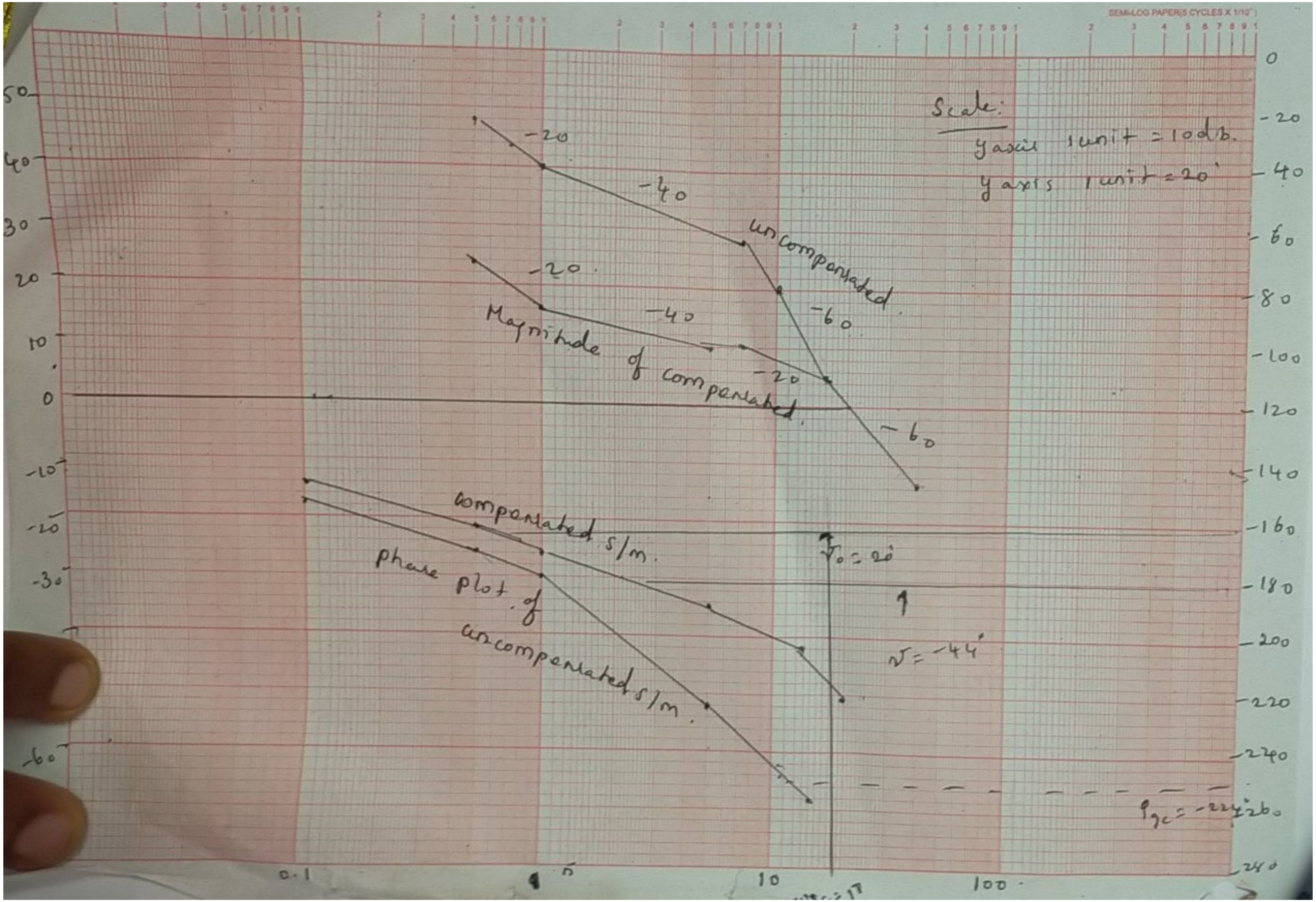
$\omega = \omega c_1$, $A_0 = 20\log\left(\frac{4}{j\omega}\right) = 12$ db

$\omega = \omega c_2$, $A_0 = -40 \times \log\frac{4.2}{1} + 12 = -13$ db

$\omega = \omega c_3$, $A_0 = 0 \times \log\frac{5}{4.2} + (-13) = -13$ db.

$\omega = \omega c_4$, $A_0 = -20 \times \log\frac{15}{5} + (-13) = -23$ db.

$\omega = \omega_h$, $A_0 = -60 \times \log\frac{30}{15} + (-23) = -41$ db.


SEMI-LOG PAPER(5 CYCLES X 1/10")
Scale:
y axis 1 unit = 10 db.
y axis 1 unit = 20°
uncompensated
Magnitude of compensated
compensated s/m.
phase plot of
uncompensated s/m.
-20
-40
-60
0.1
1
10
100
50
40
30
20
10
0
-10
-20
-30
-60
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240

Terms	corner freq	slope	change in slope
$\frac{4}{j\omega}$	-	-20	-
$\frac{1}{1+j\omega}$	$\omega_{c_1} = 1$	-20	-40
$(1+j0.24\omega)^2$	$\omega_{c_2} = 4.2$	40	$-40+40=0$
$\frac{1}{1+j0.2\omega}$	$\omega_{c_3} = \frac{1}{0.2} = 5$	-20	$0-20=-20$
$\frac{1}{(1+j0.067\omega)^2}$	$\omega_{c_4} = \frac{1}{0.067} = 15$	-40	$-20-40=-60$

phase plot:

$$\varphi_0 = \angle G_0(j\omega) = 2\tan^{-1}0.24\omega - 90^\circ - \tan^{-1}\omega - \tan^{-1}0.2\omega$$
$$-2\tan^{-1}0.067\omega.$$

ω (rad/sec)	0.1	0.5	6	2	5	10	15
$\angle G_0(j\omega)$	-94	-112	-127	-140	-150	-171	-185

$$\varphi_{gco} = -140^\circ$$

$$\gamma_0 = 180^\circ + \varphi_{gco} = 180^\circ - 140^\circ = 40^\circ.$$

the phase margin of the compensated system is satisfactory.

∴ design is accetable.

UNIT-5

STATE VARIABLE ANALYSIS.

5.1 Concept of State Variables:

The state is the condition of a system at any time t.

The set of all possible values which the state vector $x(t)$ can have at time 't' from the state space of the system.

State Variable :-

A set of variables which describes the state of the system at any time instant are called state variable.

5.2 State models for Linear & time invariant System.

State equations:

The equation which relates the derivatives of the state variables, the state variables and input is called state equation. It is given by

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

The state model of a system consist of state equation and output equations. The state model of n^{th} order system is

$$\dot{x}(t) = A\,x(t) + B\,u(t) \rightarrow \text{State equation.}$$

$$y(t) = C\,x(t) + D\,u(t) \rightarrow \text{output equation}$$

Where,

$x(t) \rightarrow$ state vector of order $[n \times 1]$

$u(t) \rightarrow$ input vector of order $(m \times 1)$

$A \rightarrow$ System Matrix

$B \rightarrow$ Input Matrix.

$y(t) \rightarrow$ output vector

$C \rightarrow$ output matrix

$D \rightarrow$ Transmission Matrix.

Limitation of state variables.

- System must be controllable.
- State variables are Measurable
- state variables are available for feedback.

5.3. Solution of state & output equation in controllable Canonical Form.

In canonical form, of a state model, the System Matrix, A will be diagonal Matrix, the canonical form of state model matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ \vdots & 0 & 0 & \cdot & & \\ 0 & 0 & 0 & 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

Advantage of state space Representation.

→ State space Analysis can be performed with initial conditions.

→. It is used for modelling & Analysis of Linear & non Linear system, time variant & time invariant System.

→ Internal states of the system can be any variable in the System.

properties of State transition Matrix

$$\rightarrow \phi(0) = e^{A \times 0} = I \text{ (unit Matrix)}$$

$$\rightarrow \phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \left[\phi(-t)\right]^{-1}.$$

$$\rightarrow \phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}.$$

$$= \phi(t_1)\, \phi(t_2)$$

5.4. Concept of controllability & observability.

A system is said to be completely observable if every state $x(t)$ can be completely identified by the measurement of the output $y(t)$ over a finite time instant.

$$Q_o = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right]$$

$|Q_o| \neq 0$, $\therefore$ It is observable.

A system is said to be completely controllable, if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_d)$ in specified finite time by a control unit $u(t)$.

$$Q_c = \left[B \,\vdots\, AB \,\vdots\, A^2 B \right].$$

$$|Q_c| \neq 0.$$

$\therefore$ the system is controllable.

STATE VARIABLE ANALYSIS - Problem

1) Obtain the state model of the system described by the following transfer function. $\frac{y(s)}{u(s)} = \frac{5}{s^2+6s+7}$.

Soln:

$$\frac{y(s)}{u(s)} = \frac{5}{s^2+6s+7}$$

$$y(s)(s^2+6s+7) = 5u(s).$$

$$s^2 y(s) + 6s y(s) + 7y(s) = 5u(s).$$

Taking inverse Laplace

$$\ddot{y} + 6\dot{y} + 7y = 5u$$

State variables

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$\dot{x}_2 = \ddot{y} = \dot{x}_2$$

$$\therefore \dot{x}_1 = \dot{y} = x_2.$$

$$\therefore \boxed{\dot{x}_1 = x_2}$$

$$\dot{x}_2 = -7x_1 - 6x_2 + 5u.$$

State Matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

Output equation

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2) Obtain the transfer function model for the following State model system $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. $D = [0]$.

Soln:

$$\frac{Y(s)}{U(s)} = C[SI - A]^{-1} B + D$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} S & -1 \\ 6 & S+5 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{1}{|SI-A|} \operatorname{adj}(SI-A)$$

$$|SI-A| = \begin{vmatrix} S & -1 \\ 6 & S+5 \end{vmatrix}$$

$$= S(S+5) + 6 = S^2 + 5S + 6$$

$$\operatorname{adj}[SI-A] = \begin{bmatrix} S+5 & 1 \\ -6 & 5 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{\begin{bmatrix} S+5 & 1 \\ -6 & 5 \end{bmatrix}}{S^2+5S+6}$$

$$= \begin{bmatrix} \frac{S+5}{S^2+5S+6} & \frac{1}{S^2+5S+6} \\ \frac{-6}{S^2+5S+6} & \frac{5}{S^2+5S+6} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D.$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+5}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{s+5}{s^2+5s+6}.$$

3) obtain the state transition matrix for the state model whose system matrix A is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Soln:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \phi(s) = L^{-1}[sI - A]^{-1}$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{1}{[SI-A]} \operatorname{adj}[SI-A]$$

$$|SI-A| = \begin{vmatrix} S-1 & -1 \\ 0 & S-1 \end{vmatrix}$$

$$= (S-1)^2$$

$$\operatorname{adj} A = \operatorname{Adj}(SI-A) = \begin{vmatrix} S-1 & 1 \\ 0 & S-1 \end{vmatrix}$$

$$[SI-A]^{-1} = \begin{bmatrix} \frac{S-1}{(S-1)^2} & \frac{1}{(S-1)^2} \\ \frac{0}{(S-1)^2} & \frac{S-1}{(S-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(S-1)} & \frac{1}{(S-1)^2} \\ 0 & \frac{1}{(S-1)} \end{bmatrix}$$

$$L^{-1}[SI-A]^{-1} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

4) A system is characterised by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$$. Identify the 1st state

as the output. determine whether or not the system is completely controllable & obserable.

$$\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$$

$$Y(s)\left(s^3+5s^2+11s+6\right) = 3\,U(s)$$

$$s^3Y(s)+5s^2Y(s)+11sY(s)+6Y(s) = 3U(s).$$

Taking inverse Laplace transform.

$$\dddot{y}+5\ddot{y}+11\dot{y}+6y = 3u$$

State variable

$$x_1 = y,\ x_2 = \dot{y} = \dot{x}_1,\ \dot{x}_2 = \ddot{y} = x_3,\ \dot{x}_3 = \dddot{y}$$

$$\dot{x}_3+5x_3+11x_2+6x_1 = 3u$$

$$\therefore\ \dot{x}_3 = 3u-5x_3-11x_2-6x_1$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_1 = x_2$$

State Matrix

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 3\end{bmatrix}u.$$

Output equation

$$y = x_1$$

output Matrix

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

<u>controllability</u>:

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$AB \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 42 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & -15 \\ 3 & -15 & 42 \end{bmatrix}$$

$$|Q_c| = 3(-9) = -27$$

$|Q_c| \neq 0$, Hence the system is controllable.

Observability:

$$Q_0 = \left[c^T \quad A^T c^T \quad (A^T)^2 c^T \right]$$

$$c^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T c^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T (A^T c^T) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|Q_0| = 1$$

$$|Q_0| \neq 0.$$

Hence the system is observable.

5) Check the controllability of the following state space system.

$$\dot{x}_1 = x_2 + u_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -2x_2 - 3x_3 + u_1 + u_2$$

Soln:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} [u_1, u_2]$$

Controllability:

$$Q_c = [B \quad AB \quad A^2B]$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -3 & -3 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \\ 7 & 7 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -3 & -3 \\ 1 & 1 & -3 & 1 & -3 & 7 & 7 \end{bmatrix}$$

$$|Q_c| = -1(-1) + [(-1)(7+9) + 1(7+9)]$$

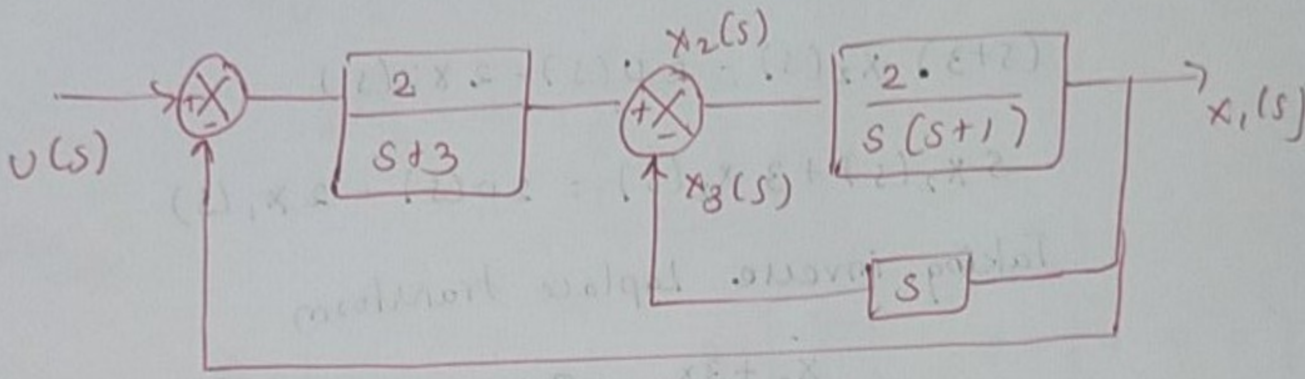
$$= 1 + [-16 + 16]$$

$$= 1$$

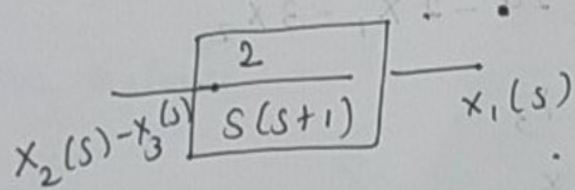
$$|Q_c| = 1$$

$|Q_c| \neq 0$ $\therefore$ System is controllable.

6) write the state equation for the system shown in fig. In which x_1, x_2, x_3 are the state vector. Determine whether system is completely controllable & observable.



<u>Soln:</u>

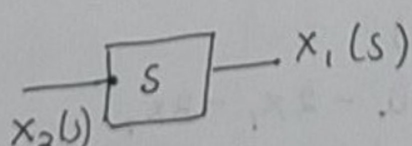


$$x_1(s) = [x_2(s) - x_3(s)]\left[\frac{2}{s(s+1)}\right].$$

$$s^2 x_1(s) + s\,x_1(s) = 2\,x_2(s) - 2x_3(s).$$

Taking inverse Laplace transform.

$$\ddot{x}_1 + \dot{x}_1 = 2x_2 - 2x_3$$

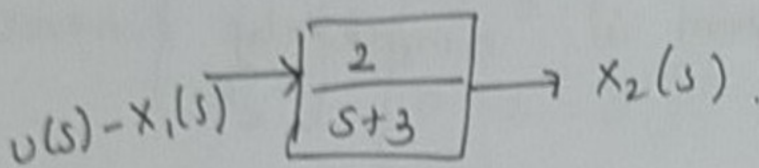


$$x_3(s) = s\,x_1(s).$$

Taking inverse Laplace transform.

$$x_3 = \dot{x}_1$$

$$X_2(s) = \left[U(s) - X_1(s)\right]\left[\frac{2}{s+3}\right]$$



$$(s+3)\, X_2(s) = 2U(s) - 2X_1(s)$$

$$s X_2(s) + 3 X_2(s) = 2U(s) - 2X_1(s)$$

Taking inverse Laplace transform

$$\dot{x}_2 + 3x_2 = 2u - 2x_1$$

$$\dot{x}_2 = 2u - 2x_1 - 3x_2$$

$$\dot{x}_1 = x_3$$

$$\ddot{x}_1 = \dot{x}_3$$

$$\dot{x}_3 + x_3 = 2x_2 - 2x_3$$

$$\dot{x}_3 = 2x_2 - 2x_3 - x_3$$

$$\boxed{\dot{x}_3 = 2x_2 - 3x_3}$$

State equation

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = 2u - 2x_1 - 3x_2$$

$$\dot{x}_3 = 2x_2 - 3x_3$$

State Matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$$

(15)

7) Check the controllability & observability of the system whose state space representation is given as.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Soln:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

By using Kalman's test,

$$Q_c = \left[B \;\vdots\; AB \;\vdots\; A^2 B \right].$$

$$B = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 21 \end{bmatrix}$$

$$A^2 B = A(AB) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -10 \\ 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 10 \\ -26 \\ -75 \end{bmatrix}.$$

$$\therefore Q_c = \begin{bmatrix} 10 & -10 & 10 \\ 1 & 8 & -26 \\ 0 & 21 & -75 \end{bmatrix}. = 10\{(8\times-7.5) - (-26\times21)\} - 1$$

$$-1\{-10\times-7.5) - (21\times10)\}$$

$$= -1080 \neq 0.$$

$\therefore$ System is controllable.

Observability

$$Q_0 = \left[C^T \;\vdots\; A^T C^T \;\vdots\; (A^T)^2 C^T \right].$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$Q_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore Q_0 = 0$. it is Not observable.